Tables of some Fourier coefficients of Hermitian modular forms of degree 2

by

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Abstract. In this paper some tables of Fourier coefficients of Hermitian modular forms of degree 2 are given. In particular Eisenstein series and their restrictions to the Siegel half space are considered.

Keywords. Fourier coefficients, Hermitian Eisenstein series, Siegel Eisenstein series, Maaß space

Mathematics Subject Classification. 11F30, 11F46, 11F55

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1 Introduction

The Hermitian modular group associated with an imaginary-quadratic number field $\mathbb{K}$ was introduced by H. Braun [1], [2] as an analogue of the Siegel modular group. In this paper we only consider the case of degree 2. Then the Fourier coefficients of the Hermitian Eisenstein series are given by an explicitly known Maaß condition [6].
As it is explicit the Fourier coefficients can be calculated by a computer exactly. The Fourier coefficients in this paper are calculated by the code in [4], Anhang B as a reference for other papers in need of explicit coefficients for low weights and discriminants. For high weights and discriminants a limit formular may be used instead of explicit calculations.

2 Eisenstein series for the Hermitian modular group

Throughout the paper let

$$\mathbb{K} = \mathbb{Q}(\sqrt{-m}) \subset \mathbb{C},\;\; m \in \mathbb{N} \text{ squarefree},$$

be an imaginary-quadratic number field with discriminant $-\Delta = -\Delta_{\mathbb{K}}$, ring of integers $\mathcal{O}_{\mathbb{K}}$ and inverse different $\mathcal{O}_{\mathbb{K}}^{\#} = \mathcal{O}_{\mathbb{K}}/\sqrt{-\Delta}$. Denote the associated primitive real Dirichlet character $\mod \Delta$ by $\chi_{\mathbb{K}} = \chi_{-\Delta}$.

Define the the *Hermitian modular group of degree* 2 by

$$\Gamma_2^{(\mathbb{K})} := \{M \in \mathcal{O}_{\mathbb{K}}^{4\times 4};\; J[M] := \overline{M}^{tr} J M = J, \det M = 1\},\;\; J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},\; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly $\Gamma_2 := \Gamma_2^{(\mathbb{K})} \cap \mathbb{R}^{4\times 4}$ is the Siegel modular group of degree 2. The *Hermitian half-space* $\mathbb{H}_2$ and the *Siegel half-space* $\mathbb{S}_2$ are defined as

$$\mathbb{H}_2 := \left\{ Z \in \mathbb{C}^{2\times 2}; \tfrac{1}{2i}(Z - \overline{Z}^{tr}) > 0 \right\} \supset \mathbb{S}_2 := \{Z \in \mathbb{H}_2; Z = Z^{tr}\}.$$

The space $\mathcal{M}_k(\Gamma_2^{(2)})$ of weight k Hermitian modular forms consists of all holomorphic functions $f : \mathbb{H}_2 \to \mathbb{C}$ satisfying

$$f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z) \text{ for all } \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2^{(\mathbb{K})}.$$

A Hermitian modular form f possesses a Fourier expansion of the form

$$f(Z) = \sum_{T \in \Lambda_2, T \geqslant 0} \alpha_f(T) \exp(2\pi i \,\mathrm{trace}(TZ)),$$

where

$$\Lambda_2 = \left\{ T = \begin{pmatrix} m & t \\ \bar{t} & m \end{pmatrix}; m, n \in \mathbb{N}_0, t \in \mathcal{O}_{\mathbb{K}}^{\#} \right\}.$$

For $T \neq 0$ let $\epsilon(T) := \max\{l \in \mathbb{N}; \frac{1}{l}T \in \Lambda_2\}$. Now we can define Hermitian Eisenstein series $E_k^{(\mathbb{K})}$ of even weight $k \geqslant 4$ according to [6], [5] as a Maaß lift via

$$E_k^{(\mathbb{K})}(Z) = 1 + \sum_{0 \neq T \in \Lambda_2, T \geqslant 0} \sum_{d | \epsilon(T)} d^{k-1} \alpha_k^*(\Delta \det T/d^2) \exp(2\pi i \operatorname{trace}(TZ)), Z \in \mathbb{H}_2,$$

where

$$\alpha_k^*(l) = \begin{cases} 0, & \text{if } l \neq 0, a_\Delta(l) = 0, \\ -\frac{2k}{B_k}, & \text{if } l = 0, \\ \frac{4k(k-1)}{B_k B_{k-1,\chi}} \frac{1}{a_\Delta(l)} \sum_{t|l} \sum_{\substack{mn=\Delta \\ \gcd(m,n)=1}} \psi_m\left(-\frac{l}{t}\right) \psi_n(t) t^{k-2}, & \text{if } l > 0, a_\Delta(l) \neq 0, \end{cases}$$

$$a_\Delta(l) = \#\{u : \mathcal{O}_\mathbb{K}^\# / \mathcal{O}_\mathbb{K} : \Delta u \overline{u} \equiv -l \pmod{\Delta}\},$$

$$\psi_m(l) = \begin{cases} \chi_\mathbb{K}(l^*), & \text{if } \gcd(l,m) = 1, l^* \equiv l \pmod{m}, l^* \equiv 1 \pmod{n}, \\ 0, & \text{else.} \end{cases}$$

and B_k, $B_{k-1,\chi}$ are the (generalized) Bernoulli numbers.
For $k > 4$ even, the Hermitian Eisenstein series is also given by the well-known absolutely convergent series

$$E_k^{(\mathbb{K})}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \backslash \Gamma_2^{(\mathbb{K})}} \det(CZ + D)^{-k}, Z \in \mathbb{H}_2.$$

3 Fourier coefficients of the identity matrix

The Fourier coefficients of the identity matrix are often used, since they are relatively easy to calculate, but non trivial.

$-\Delta$	$\alpha_{E_4^{(\mathbb{K})}}(I)$	$\alpha_{E_6^{(\mathbb{K})}}(I)$	$\alpha_{E_8^{(\mathbb{K})}}(I)$	$\alpha_{E_{10}^{(\mathbb{K})}}(I)$	$\alpha_{E_{12}^{(\mathbb{K})}}(I)$
-3	17280	120960	149760	$\frac{46759680}{809}$	$\frac{11606474880}{1276277}$
-4	14400	102816	$\frac{7862400}{61}$	$\frac{13840992}{277}$	$\frac{274810536000}{34910011}$
-7	10080	75600	$\frac{7058880}{73}$	$\frac{332917200}{8831}$	$\frac{210315661920}{35360543}$
-8	10080	$\frac{1375920}{19}$	$\frac{27961920}{307}$	$\frac{2952789840}{83579}$	$\frac{4263731166240}{765994921}$
-11	9600	$\frac{5410944}{85}$	$\frac{94483200}{1207}$	$\frac{3772716288}{124975}$	$\frac{566473353264000}{119276509631}$
-15	6240	$\frac{1577520}{31}$	$\frac{682543680}{10351}$	$\frac{21140679120}{821063}$	$\frac{4722687202353120}{1162365927091}$
-19	$\frac{86400}{11}$	$\frac{13136256}{269}$	$\frac{3226003200}{54113}$	$\frac{17037910441728}{741688727}$	$\frac{36518860110096000}{10105741955551}$
-20	6240	$\frac{26817840}{587}$	$\frac{97506240}{1693}$	$\frac{2252771392080}{100822507}$	$\frac{8945656384785120}{2541081949591}$
-23	5280	$\frac{2938320}{71}$	$\frac{986905920}{18517}$	$\frac{215355209520}{10355903}$	$\frac{2601170639104957920}{792749456479093}$
-24	$\frac{124800}{23}$	$\frac{33038208}{797}$	$\frac{13086278400}{249089}$	$\frac{5811091097856}{284922989}$	$\frac{377647197559632000}{117513534860863}$
-31	4800	$\frac{107744}{3}$	$\frac{1267862400}{27593}$	$\frac{234545035296}{13092787}$	$\frac{372930870576264000}{131949420931697}$
-35	$\frac{16640}{3}$	$\frac{42067200}{1177}$	$\frac{16341045760}{372341}$	$\frac{33027584550400}{1951561497}$	$\frac{912828487223627520}{342848496467383}$
-39	$\frac{40800}{11}$	$\frac{14395248}{457}$	$\frac{16218081600}{396617}$	$\frac{8829469334928}{552942737}$	$\frac{6060799707014052000}{2405306075520211}$
-40	$\frac{393120}{79}$	$\frac{1291988880}{39521}$	$\frac{280885475520}{6889177}$	$\frac{1730154726059760}{109493813441}$	$\frac{62457005901065499360}{25089805852970519}$
-43	$\frac{443520}{83}$	$\frac{1723075200}{53041}$	$\frac{3034254263040}{76565663}$	$\frac{3085684873286400}{202075601281}$	$\frac{9901988259904782720}{4122216597868291}$
-47	3680	$\frac{1707888}{59}$	$\frac{1710986560}{45891}$	$\frac{198427388848}{13640153}$	$\frac{23932605167302671840}{10426585217673811}$
-51	$\frac{278400}{67}$	$\frac{336760704}{11485}$	$\frac{4217316230400}{116111407}$	$\frac{1208089576835328}{86178913165}$	$\frac{354523978239487056000}{160736570936573207}$
-52	$\frac{612000}{151}$	$\frac{3670788240}{128657}$	$\frac{104259096000}{2916421}$	$\frac{14113152996695280}{1018375291937}$	$\frac{50648350317446700000}{23198161612593103}$
-55	3744	$\frac{288686160}{10687}$	$\frac{9491107392}{275113}$	$\frac{690806752835760}{51364326551}$	$\frac{16595900720826309792}{7821326688116663}$
-56	$\frac{41600}{11}$	$\frac{18354560}{671}$	$\frac{274140755200}{7963537}$	$\frac{283704081647360}{21246124181}$	$\frac{368010659143215088000}{174922434301965067}$
-59	$\frac{278400}{67}$	$\frac{407143296}{14813}$	$\frac{321375494400}{9508147}$	$\frac{234928700166912}{18023318471}$	$\frac{11162789892086008176000}{5443495277209571837}$
-67	$\frac{1077120}{251}$	$\frac{10156164480}{390241}$	$\frac{43420023440640}{1367650871}$	$\frac{107201866874952960}{8763303034081}$	$\frac{11943233295423141007296 0}{6206310363303344458 1}$
-68	$\frac{174000}{53}$	$\frac{1789041240}{72299}$	$\frac{376546092000}{12056033}$	$\frac{1183227977312040}{97644806267}$	$\frac{2098546834679527050000}{1099169373812008973}$
-71	$\frac{50400}{17}$	$\frac{53364528}{2267}$	$\frac{122000270400}{4021907}$	$\frac{355164442185168}{30007358867}$	$\frac{683601145712075052000}{366045578865601039}$
-79	$\frac{93600}{31}$	$\frac{17527536}{779}$	$\frac{7292623665600}{253362481}$	$\frac{1251614768172912}{111534497413}$	$\frac{503540137121518692000}{284410709503749473}$
-83	$\frac{110208}{31}$	$\frac{7972997760}{343841}$	$\frac{498194854656}{17481727}$	$\frac{198201716428235520}{18035105539681}$	$\frac{6161278900519560714624}{3563600333343835363}$
-84	$\frac{691200}{263}$	$\frac{2467584000}{111869}$	$\frac{84174602803200}{2998225943}$	$\frac{65428251558912000}{6001721823533}$	$\frac{52087833961750274688000}{30323013553065550063}$
-87	$\frac{67360}{27}$	$\frac{594116880}{28157}$	$\frac{1237232509760}{45189811}$	$\frac{9024445289834480}{844098195621}$	$\frac{6165313049410472237280}{3654456989580105187}$
-88	$\frac{144000}{43}$	$\frac{30229960320}{1372577}$	$\frac{222914686176000}{8109725839}$	$\frac{86312290695287040}{8101947774307}$	$\frac{182473839644267120040000 0}{108724676760234823042 9}$
-91	$\frac{39168}{11}$	$\frac{3454859520}{154877}$	$\frac{27257498058240}{1000621267}$	$\frac{124146645907299840}{11827264167629}$	$\frac{25514543968958594309760 0}{15451928820000576928 9}$
-95	$\frac{18720}{7}$	$\frac{856593360}{42013}$	$\frac{980185227840}{37374787}$	$\frac{18244108564173360}{1783003663189}$	$\frac{594382549635860600160}{368155259593308227}$

4 Fourier coefficients of the identity matrix of the restriction to the Siegel half space

The Fourier coefficients $\beta_{f|_{\mathbb{S}_2}}(R)$ of the restriction to the Siege half space are given by

$$\beta_{f|_{\mathbb{S}_2}}(R) = \sum_{\substack{T\in\Lambda_2, T\geqslant 0\\ T+\overline{T}=2R}} \alpha_f(T).$$

Only $k \geqslant 10$ is interesting, since for lower weights the space of Siegel modular forms is one dimensional.

$-\Delta$	$\beta_{E_{10}^{(\mathbb{K})}\vert_{\mathbb{S}_2}}(I)$	$\beta_{E_{12}^{(\mathbb{K})}\vert_{\mathbb{S}_2}}(I)$	$\beta_{E_{14}^{(\mathbb{K})}\vert_{\mathbb{S}_2}}(I)$
-3	$\frac{50394960}{809}$	$\frac{12008636640}{1276277}$	$\frac{426960}{611}$
-4	$\frac{16465680}{277}$	$\frac{312381407520}{34910011}$	$\frac{24352560}{41581}$
-7	$\frac{528398640}{8831}$	$\frac{300529625760}{35360543}$	$\frac{27295454160}{46095331}$
-8	$\frac{5005067760}{83579}$	$\frac{6515425818720}{765994921}$	$\frac{514323229680}{869783713}$
-11	$\frac{299276208}{4999}$	$\frac{1015185398965920}{119276509631}$	$\frac{550999753200}{931476673}$
-15	$\frac{49151052720}{821063}$	$\frac{9891151779993120}{1162365927091}$	$\frac{374164219456560}{632517358621}$
-19	$\frac{44399917180080}{741688727}$	$\frac{85998980560540320}{10105741955551}$	$\frac{298466504210160}{504550539121}$
-20	$\frac{6035531428080}{100822507}$	$\frac{21624241766793120}{2541081949591}$	$\frac{3463564363618320}{5855086886687}$
-23	$\frac{619941334320}{10355903}$	$\frac{6746157811485761760}{792749456479093}$	$\frac{668858712549840}{1130691353219}$
-24	$\frac{17056429180560}{284922989}$	$\frac{1000018305334208160}{117513534860863}$	$\frac{60892221774131280}{102936916054043}$
-31	$\frac{783777844080}{13092787}$	$\frac{1122868156803251040}{131949420931697}$	$\frac{15549762605204880}{26286533422663}$
-35	$\frac{38942386504560}{650520499}$	$\frac{2917581876405574560}{342848496467383}$	$\frac{339195287475064080}{573402231629903}$
-39	$\frac{33101012886480}{552942737}$	$\frac{20468741951609471520}{2405306075520211}$	$\frac{1046808000232965360}{1769606110093841}$
-40	$\frac{6554671640249040}{109493813441}$	$\frac{213510012465679282080}{25089805852970519}$	$\frac{13889842992609316560}{23480476300440371}$
-43	$\frac{12096921811922640}{202075601281}$	$\frac{35079366946501977120}{4122216597868291}$	$\frac{240075676460104179120}{405842695582800517}$
-47	$\frac{816545635920}{13640153}$	$\frac{88728462935889023520}{10426585217673811}$	$\frac{104607441236198160}{176836596578671}$
-51	$\frac{1031792201470800}{17235782633}$	$\frac{1367840856406140174240}{160736570936573207}$	$\frac{40518180793390302000}{68495100799859501}$
-52	$\frac{60963370733699280}{1018375291937}$	$\frac{3356010843779318044320}{394368747414082751}$	$\frac{4796589899900598518160}{8108530564632241031}$
-55	$\frac{3074841668107440}{51364326551}$	$\frac{66558160260365024160}{7821326688116663}$	$\frac{46495839344462482320}{78600202410394087}$
-56	$\frac{1271864671917840}{21246124181}$	$\frac{1488560193487354609440}{174922434301965067}$	$\frac{2196038311219096560}{3712354867389601}$
-59	$\frac{1078936469457840}{18023318471}$	$\frac{46323218213856527055840}{5443495277209571837}$	$\frac{516744612683067965040}{873545495723782649}$
-67	$\frac{524600905606802640}{8763303034081}$	$\frac{528146447888047017949920}{62063103633033444581}$	$\frac{797757422968918942648560}{1348591522210277501521}$
-68	$\frac{5845347366954480}{97644806267}$	$\frac{9353744292627563643360}{1099169373812008973}$	$\frac{586187950870018547280}{990937945329579323}$
-71	$\frac{1796341633012080}{30007358867}$	$\frac{3114985602499094328480}{366045578865601039}$	$\frac{326844527055939462960}{552523543513948781}$
-79	$\frac{6676830933307920}{111534497413}$	$\frac{2420286732920754603360}{284410709503749473}$	$\frac{3008272377028867004880}{5085418835324120963}$
-83	$\frac{1079642206964114640}{18035105539681}$	$\frac{30325632508301782448160}{3563600333343835363}$	$\frac{184107707520962871213360}{311230063609683311501}$
-84	$\frac{359283294492811920}{6001721823533}$	$\frac{258043684975649783552160}{30323013553065550063}$	$\frac{1347349235311799924508720}{2277664493093829972317}$
-87	$\frac{16843521313468080}{281366065207}$	$\frac{31098807195252539127840}{3654456989580105187}$	$\frac{4783548146862341599920}{8086483799378774197}$
-88	$\frac{485009903491732080}{8101947774307}$	$\frac{9252285015393331415693280}{1087246767602348230429}$	$\frac{1853868310953121581931440}{3133923941854539265829}$
-91	$\frac{708019899986543760}{11827264167629}$	$\frac{1314932854839914817168480}{154519288200005769289}$	$\frac{333089190958924834396560}{563080011761268776071}$
-95	$\frac{15248086980896880}{254714809027}$	$\frac{3132938625858043460640}{368155259593308227}$	$\frac{33518271611922108718320}{56661907037343777937}$

For growing weights we have the following table with numerical approximations of the same coefficients. An approximation is used due to space constraints on this page, all calculations until the final output were done exact.

k	$\Delta = 3$	$\Delta = 4$	$\Delta = 7$
10	62292.9048207664	59442.8880866426	59834.5193069868
12	9409.11466711380	8948.18989085967	8499.01048634915
14	698.788870703764	585.665568408648	592.152254205529
16	28.2539302924599	34.1914584280348	22.6150549414149
18	0.671999029810269	-0.717949911093484	0.513048780524854
20	0.00999093956855868	0.159931349063353	0.00734648301028152
22	0.0000974914248661370	-0.0141266891394328	0.0000696300579152494
24	$6.49752962137788 \times 10^{-7}$	0.00110936242636458	$4.53880913025321 \times 10^{-7}$
26	$3.05712298468136 \times 10^{-9}$	-0.0000729533261224574	$2.10010633628718 \times 10^{-9}$
28	$1.04412681070186 \times 10^{-11}$	$4.10283900187409 \times 10^{-6}$	$7.08345565666433 \times 10^{-12}$
30	$2.65083170040704 \times 10^{-14}$	$-1.99474403837109 \times 10^{-7}$	$1.78167036283595 \times 10^{-14}$
32	$5.10615231301220 \times 10^{-17}$	$8.46767179522254 \times 10^{-9}$	$3.40829980311166 \times 10^{-17}$
34	$7.59687764706780 \times 10^{-20}$	$-3.16562766582345 \times 10^{-10}$	$5.04497739143897 \times 10^{-20}$
36	$8.86775464579238 \times 10^{-23}$	$1.05020143700533 \times 10^{-11}$	$5.86676621366158 \times 10^{-23}$
38	$8.23489137560673 \times 10^{-26}$	$-3.11263445187179 \times 10^{-13}$	$5.43293560039954 \times 10^{-26}$
40	$6.15947366957965 \times 10^{-29}$	$8.29162501626647 \times 10^{-15}$	$4.05536701481774 \times 10^{-29}$
42	$3.75227389460431 \times 10^{-32}$	$-1.99597704275077 \times 10^{-16}$	$2.46675432120954 \times 10^{-32}$
44	$1.88045553487672 \times 10^{-35}$	$4.36312376646947 \times 10^{-18}$	$1.23484488410187 \times 10^{-35}$
46	$7.82341197179212 \times 10^{-39}$	$-8.69945566223161 \times 10^{-20}$	$5.13323376034675 \times 10^{-39}$
48	$2.72451767715640 \times 10^{-42}$	$1.58853257892773 \times 10^{-21}$	$1.78658562349500 \times 10^{-42}$
50	$8.00262906162917 \times 10^{-46}$	$-2.66635852589439 \times 10^{-23}$	$5.24536001600470 \times 10^{-46}$
52	$1.99639233541531 \times 10^{-49}$	$4.12798491639990 \times 10^{-25}$	$1.30812032337893 \times 10^{-49}$
54	$4.25708805690760 \times 10^{-53}$	$-5.91314631328461 \times 10^{-27}$	$2.78875875702656 \times 10^{-53}$
56	$7.80559525379893 \times 10^{-57}$	$7.85998853574130 \times 10^{-29}$	$5.11244083561156 \times 10^{-57}$
58	$1.23740510104833 \times 10^{-60}$	$-9.72117511840404 \times 10^{-31}$	$8.10360548839283 \times 10^{-61}$
60	$1.70470219727681 \times 10^{-64}$	$1.12149798635138 \times 10^{-32}$	$1.11628181544858 \times 10^{-64}$
62	$2.05061271969510 \times 10^{-68}$	$-1.20969852042694 \times 10^{-34}$	$1.34269944018967 \times 10^{-68}$
64	$2.16346242551063 \times 10^{-72}$	$1.22265702418812 \times 10^{-36}$	$1.41651883021842 \times 10^{-72}$
66	$2.01027270635281 \times 10^{-76}$	$-1.16030203336056 \times 10^{-38}$	$1.31616908540254 \times 10^{-76}$
68	$1.65157545230088 \times 10^{-80}$	$1.03588621031645 \times 10^{-40}$	$1.08129243190691 \times 10^{-80}$
70	$1.20413775550017 \times 10^{-84}$	$-8.71593103185594 \times 10^{-43}$	$7.88337392903792 \times 10^{-85}$
72	$7.81793630472573 \times 10^{-89}$	$6.92336348262348 \times 10^{-45}$	$5.11825149334774 \times 10^{-89}$
74	$4.53487582388687 \times 10^{-93}$	$-5.20021755695906 \times 10^{-47}$	$2.96886293777570 \times 10^{-93}$
76	$2.35742187955366 \times 10^{-97}$	$3.69903351976831 \times 10^{-49}$	$1.54332919652203 \times 10^{-97}$
78	$1.10148252195545 \times 10^{-101}$	$-2.49542019866592 \times 10^{-51}$	$7.21101353992347 \times 10^{-102}$
80	$4.63863351853238 \times 10^{-106}$	$1.59875431193252 \times 10^{-53}$	$3.03673490028708 \times 10^{-106}$
82	$1.76529565041447 \times 10^{-110}$	$-9.74016826747515 \times 10^{-56}$	$1.15566746606081 \times 10^{-110}$
84	$6.08619844219320 \times 10^{-115}$	$5.64981531589913 \times 10^{-58}$	$3.98437795863758 \times 10^{-115}$
86	$1.90550455718544 \times 10^{-119}$	$-3.12389031412550 \times 10^{-60}$	$1.24745148928638 \times 10^{-119}$

5 Cusp forms of weight 10 and 12

For weight 10 resp. 12 there exists a unique normalized Siegel cusp form. The following coefficients show that these are also the restrictions of Hermitian cusp forms.

$-\Delta$	$\beta_{(E_{10}^{(\mathbb{K})}-E_4^{(\mathbb{K})}E_6^{(\mathbb{K})})\vert_{\mathbb{S}_2}}(I)$	$\beta_{(E_{12}^{(\mathbb{K})}-\frac{441}{691}(E_4^{(\mathbb{K})})^3-\frac{250}{691}(E_6^{(\mathbb{K})})^2)\vert_{\mathbb{S}_2}}(I)$
-3	$\frac{87091200}{809}$	$-\frac{731566080000}{1276277}$
-4	$\frac{29030400}{277}$	$-\frac{20026621440000}{34910011}$
-7	$\frac{928972800}{8831}$	$-\frac{20300958720000}{35360543}$
-8	$\frac{8796211200}{83579}$	$-\frac{439762659840000}{765994921}$
-11	$\frac{526030848}{4999}$	$-\frac{68476779786240000}{119276509631}$
-15	$\frac{86394470400}{821063}$	$-\frac{667317569264640000}{1162365927091}$
-19	$\frac{78042917836800}{741688727}$	$-\frac{5801730886103040000}{10105741955551}$
-20	$\frac{10608840345600}{100822507}$	$-\frac{1458841433495040000}{2541081949591}$
-23	$\frac{1089685094400}{10355903}$	$-\frac{455119450099614720000}{792749456479093}$
-24	$\frac{29980535961600}{284922989}$	$-\frac{67464816041387520000}{117513534860863}$
-31	$\frac{1377666662400}{13092787}$	$-\frac{75752491408634880000}{131949420931697}$
-35	$\frac{68449996339200}{650520499}$	$-\frac{196830179472568320000}{342848496467383}$
-39	$\frac{58182495436800}{552942737}$	$-\frac{1380892234965949440000}{2405306075520211}$
-40	$\frac{11521311017932800}{109493813441}$	$-\frac{14404120233691829760000}{25089805852970519}$
-43	$\frac{21263071086028800}{202075601281}$	$-\frac{2366574849723248640000}{4122216597868291}$
-47	$\frac{1435262976000}{13640153}$	$-\frac{5985928644223933440000}{10426585217673811}$
-51	$\frac{1813607301703680}{17235782633}$	$-\frac{92279267246182625280000}{160736570936573207}$
-52	$\frac{107156873975961600}{1018375291937}$	$-\frac{226408083949199255040000}{394368747414082751}$
-55	$\frac{5404727520460800}{51364326551}$	$-\frac{4490243204854901760000}{7821326688116663}$
-56	$\frac{2235588864768000}{21246124181}$	$-\frac{100423406816858119680000}{174922434301965067}$
-59	$\frac{1896474195302400}{18023318471}$	$-\frac{3125124248348876820480000}{5443495277209571837}$
-67	$\frac{922104331232716800}{8763303034081}$	$-\frac{35630582964090542346240000}{62063103633033444581}$
-68	$\frac{10274515779225600}{97644806267}$	$-\frac{631035885600907153920000}{1099169373812008973}$
-71	$\frac{3157475431219200}{30007358867}$	$-\frac{210147681970695490560000}{366045578865601039}$
-79	$\frac{11736035735961600}{111534497413}$	$-\frac{163280899395664081920000}{284410709503749473}$
-83	$\frac{1897714594244044800}{18035105539681}$	$-\frac{2045871860735211909120000}{3563600333343835363}$
-84	$\frac{631521396408268800}{6001721823533}$	$-\frac{17408517892480965035520000}{30323013553065550063}$
-87	$\frac{29606286031257600}{281366065207}$	$-\frac{2098032894230138772480000}{3654456989580105187}$
-88	$\frac{852514254534297600}{8101947774307}$	$-\frac{624191087495176096412160000}{1087246767602348230429}$
-91	$\frac{1244504602630195200}{11827264167629}$	$-\frac{887099096665984449945600000}{1545192882000005769289}$
-95	$\frac{26801950718361600}{254714809027}$	$-\frac{211358854951570529280000}{368155259593308227}$

6 Difference between the restriction and Siegel Eisenstein series

You can also construct a Siegel cusp form by the difference between the restriction of the Hermitian Eisenstein series and the Siegel Eisenstein series S_k. The Fourier coefficient of the identity matrix shows that this cusp form is not vanishing identically. Due to [7] the relevant coefficient of the Siegel Eisenstein series is

$$\beta_{S_k}(I) = -\frac{4k}{B_k}\frac{B_{k-1,\chi_{-4}}}{B_{2k-2}}.$$

$-\Delta$	$\beta_{E_{10}^{(\mathbb{K})}\vert_{\mathbb{S}_2}-S_{10}}(I)$	$\beta_{E_{12}^{(\mathbb{K})}\vert_{\mathbb{S}_2}-S_{12}}(I)$	$\beta_{E_{14}^{(\mathbb{K})}\vert_{\mathbb{S}_2}-S_{14}}(I)$
-3	$\frac{86220288000}{35488403}$	$\frac{89159616000000}{99145026191}$	$\frac{43110144000}{401995841}$
-4	$-\frac{5109350400}{12151159}$	$\frac{1188794880000000}{2711914384513}$	$-\frac{160944537600}{27357428911}$
-7	$-\frac{11176704000}{387389477}$	$-\frac{29719872000000}{2746913061869}$	$\frac{18307441152000}{30327547220161}$
-8	$\frac{76640256000}{3666359993}$	$-\frac{237758976000000}{59504783448043}$	$-\frac{129023870976000}{572257668077803}$
-11	$\frac{843042816}{219291133}$	$\frac{12630945600000000}{9265757097664973}$	$-\frac{9188591892480}{612847378943563}$
-15	$-\frac{23950080000}{36017570621}$	$-\frac{29868471360000000}{90296072314210153}$	$-\frac{353072079360000}{416152778274873151}$
-19	$-\frac{3070080921600}{32535659387309}$	$\frac{64937920320000000}{785044352333068333}$	$\frac{224128593561600}{3319594407544418651}$
-20	$-\frac{1916006400000}{4422780914569}$	$\frac{5183145676800000}{197398869090077653}$	$-\frac{2677618944000000}{3852243170444864597}$
-23	$\frac{91010304000}{454282396901}$	$-\frac{402615105984000000}{6158315602766538151 9}$	$-\frac{112594116096000}{743916892714729889}$
-24	$-\frac{935011123200}{12498716758463}$	$-\frac{146221770240000000}{912880392859642042 9}$	$\frac{22039767971020800}{6772538811635256503 3}$
-31	$-\frac{22672742400}{574341287329}$	$\frac{80392253760000000}{1025022686623701805 1}$	$\frac{419726110003200}{1729472522130609025 3}$
-35	$\frac{1612638720000}{28536382729633}$	$-\frac{38487234240000000}{266334997510757135 89}$	$\frac{1193504336640000}{377259103658493710693}$
-39	$-\frac{136994457600}{24255939043979}$	$-\frac{272382626880000000}{186851391864636551113}$	$-\frac{7572005212262400}{116427871762015090297 1}$
-40	$\frac{127350558720000}{4803165114216347}$	$\frac{260227199232000000 0}{194905138807630882747 7}$	$\frac{117915440670720000}{1544853325282503373240 1}$
-43	$-\frac{210802217472000}{8864450401393627}$	$\frac{393292210752000000}{320226151972202449753}$	$-\frac{481193286190848000}{2670164905474875269503 27}$
-47	$\frac{4441651200}{598352591651}$	$-\frac{253361908800000000}{809968419464554659913}$	$\frac{410892043161600}{116346278823601589701}$
-51	$\frac{6904137461760}{756082076761811}$	$-\frac{1098892267200000000}{1248649904006581643938 1}$	$\frac{7256228737443840}{4506505016435236135243 1}$
-52	$-\frac{299229106176000}{44673068931400379}$	$-\frac{7600441185792000000}{3063574740536819034593 3}$	$\frac{133728713977098240000}{5334853622919054973766861}$
-55	$-\frac{2195424000000}{2253198912812717}$	$\frac{687016449100800000}{607584121112966731829}$	$-\frac{78011398080000000}{5171350977207299205399 7}$
-56	$\frac{2866345574400}{932003729447927}$	$\frac{3054014046720000000}{1358849946387955229976 1}$	$-\frac{1516385293516800}{2442473350256507575531}$
-59	$-\frac{5456612044800}{790628911367357}$	$-\frac{33048646263360000000}{422867043619471169013671}$	$\frac{155199514076006400}{5747326615470440420392 19}$
-67	$\frac{2106492562944000}{384419814196031227}$	$-\frac{346381928133696000000}{4821248079524937075385823}$	$\frac{1552467217433932800 0}{8872801687993300868532130 51}$
-68	$\frac{6054580224000}{4283384716514489}$	$-\frac{4902827844096000000}{853867744658382930495 59}$	$-\frac{292590693255168000}{6519687933086354535607 13}$
-71	$-\frac{564263884800}{1316332811418689}$	$\frac{4858398921600000000}{284355187030164855126 37}$	$\frac{38132506905292800}{3635223675076758354321 11}$
-79	$-\frac{1088610969600}{4892683798016071}$	$-\frac{11352354249600000000}{220938771463797703110 59}$	$-\frac{246511385029324800}{3345854699743634229307553}$
-83	$-\frac{23902179840000}{791145974709186427}$	$-\frac{6733750278528000000}{2768311646951491625039 29}$	$\frac{2340291980138112000 0}{2047679069807825508191644 31}$
-84	$-\frac{154675364659200}{263277531232922111}$	$-\frac{87651035297280000000}{2355582661842791125544029}$	$-\frac{7152349644668067840 0}{14985460776057166475164961 27}$
-87	$\frac{11650915584000}{12342685182435469}$	$-\frac{5990634599040000000}{283889182321551311241721}$	$\frac{225259210649088000}{53203483726090762862064 07}$
-88	$\frac{41564565504000}{355408143015525169}$	$\frac{7733317544709120000 0}{8446059064765321758441600 7}$	$-\frac{5603543438666342400 0}{2061905712988298873706139799}$
-91	$\frac{62560802304000}{518826597241381343}$	$\frac{1982648027767680000 0}{1200352186524104817567738 7}$	$-\frac{1346779169292211200 0}{3704677952181033271091691 01}$
-95	$-\frac{1762041600000}{11173574527587409}$	$\frac{432430081574400000}{285994050309869629980 41}$	$-\frac{1121724749376000000}{3727962515898662916186834 7}$

We describe the same table for growing weights as numerical approximations.

k	$\Delta = 3$	$\Delta = 4$	$\Delta = 7$
10	2429.53417768616	-420.482556437620	-28.8513360934685
12	899.284809590313	438.360033336186	-10.8193711743387
14	107.240273662433	-5.88302863268290	0.603657164197890
16	5.82497493657534	11.7625030721502	0.186099585530249
18	0.169484087685264	-1.22046485321849	0.0105338383998490
20	0.00291744838235833	0.152857857877152	0.000272991824081179
22	0.0000318291630556106	-0.0141923514012433	$3.96779610472303 \times 10^{-6}$
24	$2.31668627906846 \times 10^{-7}$	0.00110894434203035	$3.57965787943785 \times 10^{-8}$
26	$1.17087759835907 \times 10^{-9}$	-0.0000729552123678437	$2.13860949964895 \times 10^{-10}$
28	$4.24408251785333 \times 10^{-12}$	$4.10283280468850 \times 10^{-6}$	$8.86270067499016 \times 10^{-13}$
30	$1.13315106145211 \times 10^{-14}$	$-1.99474419013916 \times 10^{-7}$	$2.63989723881026 \times 10^{-15}$
32	$2.27935243612849 \times 10^{-17}$	$8.46767176695454 \times 10^{-9}$	$5.81499926227951 \times 10^{-18}$
34	$3.52160467425755 \times 10^{-20}$	$-3.16562766623098 \times 10^{-10}$	$9.69704418628721 \times 10^{-21}$
36	$4.24963357687507 \times 10^{-23}$	$1.05020143700071 \times 10^{-11}$	$1.24864514474427 \times 10^{-23}$
38	$4.06466774093022 \times 10^{-26}$	$-3.11263445187221 \times 10^{-13}$	$1.26271196572304 \times 10^{-26}$
40	$3.12181747846533 \times 10^{-29}$	$8.29162501626644 \times 10^{-15}$	$1.01771082370342 \times 10^{-29}$
42	$1.94775213045391 \times 10^{-32}$	$-1.99597704275077 \times 10^{-16}$	$6.62232557059149 \times 10^{-33}$
44	$9.97524413321304 \times 10^{-36}$	$4.36312376646947 \times 10^{-18}$	$3.51913762546456 \times 10^{-36}$
46	$4.23310286772920 \times 10^{-39}$	$-8.69945566223161 \times 10^{-20}$	$1.54292465628383 \times 10^{-39}$
48	$1.50122246880897 \times 10^{-42}$	$1.58853257892773 \times 10^{-21}$	$5.63290415147580 \times 10^{-43}$
50	$4.48396275487928 \times 10^{-46}$	$-2.66635852589439 \times 10^{-23}$	$1.72669370925480 \times 10^{-46}$
52	$1.13607079181316 \times 10^{-49}$	$4.12798491639990 \times 10^{-25}$	$4.47798779776779 \times 10^{-50}$
54	$2.45765984118013 \times 10^{-53}$	$-5.91314631328461 \times 10^{-27}$	$9.89330541299081 \times 10^{-54}$
56	$4.56707016467371 \times 10^{-57}$	$7.85998853574130 \times 10^{-29}$	$1.87391574648634 \times 10^{-57}$
58	$7.33135921006625 \times 10^{-61}$	$-9.72117511840404 \times 10^{-31}$	$3.06091368797576 \times 10^{-61}$
60	$1.02192629758720 \times 10^{-64}$	$1.12149798635138 \times 10^{-32}$	$4.33505915758974 \times 10^{-65}$
62	$1.24292360323095 \times 10^{-68}$	$-1.20969852042694 \times 10^{-34}$	$5.35010323725520 \times 10^{-69}$
64	$1.32501404115118 \times 10^{-72}$	$1.22265702418812 \times 10^{-36}$	$5.78070445858965 \times 10^{-73}$
66	$1.24331920390063 \times 10^{-76}$	$-1.16030203336056 \times 10^{-38}$	$5.49215582950355 \times 10^{-77}$
68	$1.03098252040389 \times 10^{-80}$	$1.03588621031645 \times 10^{-40}$	$4.60699500009915 \times 10^{-81}$
70	$7.58302969908523 \times 10^{-85}$	$-8.71593103185594 \times 10^{-43}$	$3.42502607312143 \times 10^{-85}$
72	$4.96453330310193 \times 10^{-89}$	$6.92336348262348 \times 10^{-45}$	$2.26484849172393 \times 10^{-89}$
74	$2.90263895945023 \times 10^{-93}$	$-5.20021755695906 \times 10^{-47}$	$1.33662607333906 \times 10^{-93}$
76	$1.52034362368221 \times 10^{-97}$	$3.69903351976831 \times 10^{-49}$	$7.06250940650575 \times 10^{-98}$
78	$7.15495386848896 \times 10^{-102}$	$-2.49542019866592 \times 10^{-51}$	$3.35114218885789 \times 10^{-102}$
80	$3.03391357476687 \times 10^{-106}$	$1.59875431193252 \times 10^{-53}$	$1.43201495652157 \times 10^{-106}$
82	$1.16220762050460 \times 10^{-110}$	$-9.74016826747515 \times 10^{-56}$	$5.52579436150945 \times 10^{-111}$
84	$4.03221622128041 \times 10^{-115}$	$5.64981531589913 \times 10^{-58}$	$1.93039573772479 \times 10^{-115}$
86	$1.27006425322256 \times 10^{-119}$	$-3.12389031412550 \times 10^{-60}$	$6.12011185323501 \times 10^{-120}$

7 Products of weight 18 and discriminant -3

Let $\Delta = 3$. The following 5 modular forms are all distinct products of Eisenstein series of weight 18, since in this case $(E_4^{(\mathbb{K})})^2 = E_8^{(\mathbb{K})}$, see [3] Theorem 6.

f	$E_{18}^{(\mathbb{K})}$	$E_{12}^{(\mathbb{K})} E_6^{(\mathbb{K})}$	$E_{10}^{(\mathbb{K})} (E_4^{(\mathbb{K})})^2$	$E_6^{(\mathbb{K})} (E_4^{(\mathbb{K})})^3$	$(E_6^{(\mathbb{K})})^3$
$\alpha_f(0)$	1	1	1	1	1
$\alpha_f\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)$	$-\frac{28728}{43867}$	$-\frac{282744}{691}$	216	216	-1512
$\alpha_f\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$	$\frac{3709938516480}{5537559562267}$	$\frac{44001377280}{1276277}$	$-\frac{37117440}{809}$	-207360	1886976
$\alpha_f\left(\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\right)$	$\frac{243130831973653680}{5537559562267}$	$-\frac{103949130463920}{1276277}$	$-\frac{55774934640}{809}$	-144030960	-693980784
$\alpha_f\left(\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)\right)$	$\frac{1593430773669291158784 0}{5537559562267}$	$-\frac{6111630538191360}{1276277}$	$\frac{4564637038080}{809}$	-14165383680	202674286080

With these coefficients you can determine that the modular forms are linearly independent.

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