Lehrstuhl A für Mathematik
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10. Übung zu Zahlbereichserweiterungen

(Abgabe: Montag, 14.07.2003, vor der Übung)

Aufgabe 1: Bestimmen Sie alle Ringhomomorphismen $\varphi : \mathbb{Q} \to \mathbb{R}$ und $\psi : \mathbb{R} \to \mathbb{Q}$. 5

Aufgabe 2: Es sei G die Menge aller Abbildungen $f : \mathbb{Z} \to \mathbb{Z}$, zu denen es eine natürliche Zahl c gibt, so dass für alle $x, y \in \mathbb{Z}$ gilt

$$|f(x+y) - f(x) - f(y)| \leq c.$$

Für $f, g \in G$ sei $f \oplus g : \mathbb{Z} \to \mathbb{Z}$ definiert durch $(f \oplus g)(x) := f(x) + g(x)$ für alle $x, y \in \mathbb{Z}$. Zeigen Sie:

a) $(G, \oplus)$ ist eine kommutative Gruppe. 2

b) $|f(mn) - nf(m)| \leq (n-1)c$ für alle $f \in G$ und $m, n \in \mathbb{N}$. 3

c) Für jedes $f \in G$ existiert
$$\Phi(f) := \lim_{n\to\infty} \frac{f(n)}{n} \in \mathbb{R}.$$ 3

d) Die in c) angegebene Abbildung liefert einen surjektiven Gruppenhomomorphismus $\Phi : (G, \oplus) \to (\mathbb{R}, +)$.
Hinweis: Für $r \in \mathbb{R}$ betrachte man die Abbildung $\mathbb{Z} \to \mathbb{Z}, x \mapsto [rx]$. 4

Aufgabe 3: Eine Folge $(a_n)_n$ sei definiert durch $a_1 := \alpha$ mit $0 \leq \alpha \leq \frac{1}{2}$ und $a_{n+1} := a_n^2 + \frac{1}{4}$ für alle $n \in \mathbb{N}$. Zeigen Sie:

a) $(a_n)_n$ ist monoton wachsend. 1

b) Für alle $n \in \mathbb{N}$ ist $0 \leq a_n \leq \frac{1}{2}$. 1

c) $(a_n)_n$ konvergiert. 1

d) Bestimmen Sie $\lim_{n\to\infty} a_n$. 2

Aufgabe 4: Bestimmen Sie die g-adische Darstellung der folgenden Zahlen:

a) $\frac{17}{6}$ für $g = 8$, 1

b) $\frac{6}{7}$ für $g = 5$. 1

Aufgabe 5: Bestimmen Sie alle $g \in \mathbb{N}$, $g \geq 2$ für die $\sum_{k=0}^{4} g^k$ ein Quadrat aus $\mathbb{Z}$ ist. 4

Aufgabe 6:

a) Sei $g \in \mathbb{N}$, $g \geq 2$ und $x \in \mathbb{R}$. Zeigen Sie die Äquivalenz der beiden folgenden Aussagen:

(i) x besitzt eine endliche g-adische Darstellung, also

$$x = \varepsilon \sum_{k=-N}^{M} a_k g^{-k} \text{ mit } \varepsilon = \pm 1,\ a_k \in \{0, \dots, g-1\},\ N, M \in \mathbb{N}.$$

(ii) Es existieren teilerfremde $a, b \in \mathbb{Z}$, $b \neq 0$ und ein $m \in \mathbb{N}$ mit $x = \frac{a}{b} \in \mathbb{Q}$ und $b | g^m$. 4

b) Für welche Grundzahlen $g \geq 2$ existiert eine endliche g-adische Darstellung aller Stammbrüche $\frac{1}{b}$, $b = 2, \dots, 12$? 2

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Augustin Louis Cauchy[1]

Born: 21 Aug 1789 in Paris
Died: 23 May 1857 in Sceaux (near Paris)

Paris was a difficult place to live in when Augustin-Louis Cauchy was a young child due to the political events surrounding the French Revolution. Cauchy's father was active in the education of young Augustin-Louis. Laplace and Lagrange were visitors at the Cauchy family home and Lagrange in particular seems to have taken an interest in young Cauchy's mathematical education. Lagrange advised Cauchy's father that his son should obtain a good grounding in languages before starting a serious study of mathematics. In 1802 Augustin-Louis entered the Ecole Centrale du Panthéon where he spent two years studying classical languages.

From 1804 Cauchy attended classes in mathematics and he took the entrance examination for the Ecole Polytechnique in 1805. He was examined by Biot and placed second. At the Ecole Polytechnique he attended courses by Lacroix, de Prony and Hachette while his analysis tutor was Ampère. In 1807 he graduated from the Ecole Polytechnique and entered the engineering school Ecole des Ponts et Chaussées. He was an outstanding student and for his practical work he was assigned to the Ourcq Canal project where he worked under Pierre Girard.

In 1810 Cauchy took up his first job in Cherbourg to work on port facilities for Napoleon's English invasion fleet. He took a copy of Laplace's *Méchanique Céleste* and one of Lagrange's *Thèorie des Fonctions* with him. It was a busy time for Cauchy, writing home about his daily duties he said:

> *I get up at four o'clock each morning and I am busy from then on. ... I do not get tired of working, on the contrary, it invigorates me and I am in perfect health...*

An academic career was what Cauchy wanted and he applied for a post in the Bureau des Longitudes. He failed to obtain this post, Legendre being appointed. He also failed to be appointed to the geometry section of the Institute, the position going to Poinsot. Cauchy obtained further sick leave, having unpaid leave for nine months, then political events prevented work on the Ourcq Canal so Cauchy was able to devote himself entirely to research for a couple of years.

Other posts became vacant but one in 1814 went to Ampère and a mechanics vacancy at the Institute, which had occurred when Napoleon Bonaparte resigned, went to Molard. In this last election Cauchy did not receive a single one of the 53 votes cast. His mathematical output remained strong and in 1814 he published the **memoir on definite integrals that later became the basis of his theory of complex functions**. (Fortsetzung folgt)

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland.