TENSOR PRODUCTS AND FORGETFUL FUNCTORS OF ENTWINED MODULES

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ABSTRACT. Let A and B be a K-algebras and let $\mathcal{M}$ be the category of K-modules. It is well known that any algebra-homomorphism $f : A \longrightarrow B$ induces a forgetful functor $U_f : \mathcal{M}_B \longrightarrow \mathcal{M}_A$ that commutes with the two underlying functors $U_A : \mathcal{M}_A \longrightarrow \mathcal{M}$ and $U_B : \mathcal{M}_B \longrightarrow \mathcal{M}$ resp. i.e. $U_A U_f = U_B$. Conversely any functor $\mathcal{F} : \mathcal{M}_B \longrightarrow \mathcal{M}_A$ satisfying $U_A U_f = U_B$ is of the form $\mathcal{F} = U_f$ for a uniquely determined algebra-homomorphism $f : A \longrightarrow B$ [**?**].

Similarly there is a bijection between the monoidal (tensor) stuctures on $\mathcal{M}_A$ and $U_A : \mathcal{M}_A \longrightarrow \mathcal{M}$ and the bialgebra structures on A.

In this paper we extend these facts to the category $\mathcal{M}_A^C(\psi)$ of entwined modules over an algebra A and a coalgebra C of an entwined structure (A, C, ψ). This leads to the definition of morphisms $(f, g) : (A, C, \psi) \longrightarrow (A', C', \psi')$ where $f : C' \otimes A \longrightarrow A'$ is an entwined measuring of algebras and $g : C' \longrightarrow A' \otimes C$ is an entwined comeasuring of coalgebras. In the case of tensor categories and functors on $\mathcal{M}_A^C(\psi)$ we obtain as diagonal on A an entwined double measuring $\widetilde{\Delta}_A : C \otimes C \otimes A \longrightarrow A \otimes A$ and as multiplication on C an entwined double comeasuring $\widetilde{\nabla}_C : C \otimes C \longrightarrow A \otimes A \otimes C$ satisfying certain compatibility conditions.

INTRODUCTION

In [**?**] and [**?**] the notion of an entwined module was introduced and studied under the auspices of Galois theory. In [**?**] we studied the structure of the category of entwined modules.

Let C be a coalgebra and A be an algebra. Let $\psi : C \otimes A \longrightarrow A \otimes C$ be an entwining, i.e. a linear map satisfying the axioms of a certain distributivity [**?**, **?**] given in section 1. Such a triple (A, C, ψ) is called an entwined structure. Then we can define an entwined module P to be a right C-comodule and a right A-module such that the two structures are compatible, i.e. the comodule structure map is a homomorphism of A-modules and the module structure morphism is a homomorphism of C-comodules. Morphisms of entwined modules are linear maps which are simultaneously homomorphisms of A-modules and of C-comodules. So we get the category of entwined modules $\mathcal{M}_A^C(\psi)$. They generalize e.g. Doi-Hopf modules [**?**].

If A and C are bialgebras then the tensor product of A-modules as well as of C-comodules is again an A-module resp. a C-comodule. In [**?**] we studied necessary and sufficient conditions for the entwined structure (A, C, ψ) so that the category of entwined modules with this tensor product becomes a monoidal category. We also studied conditions for this category to become a braided category.

This is, however, not the most general situation. We will investigate under which conditions the category of entwined modules becomes a monoidal category such that the underlying functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ to the category of vector spaces preserves the tensor product. In

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this paper we will develop necessary and sufficient conditions on (A, C, ψ) for $\mathcal{M}_A^C(\psi)$ and $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ to be monoidal.

In [?] Brzeziński defined and studied morphisms between entwined structures (A, C, ψ) and (A', C', ψ'). They were defined as a pair of morphisms $f : A \longrightarrow A'$, $g : C \longrightarrow C'$ satisfying certain compatibility conditions. These morphisms induced some interesting new functors on entwined modules.

In this paper we require that a morphism of entwined structures $\Phi : (A, C, \psi) \longrightarrow (A', C', \psi')$ induces a forgetful $U_\Phi : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ (or $U_\Phi : \mathcal{M}_{A'}^{C'}(\psi') \longrightarrow \mathcal{M}_A^C(\psi)$) that is compatible with the underlying functors to the category of vector spaces, or more generally $\mathbb{K}$-modules, $\mathcal{M}$. This is not the case for Brzezinski's definition of morphisms of entwined structures. So we will get a different definition of such a morphism in this paper.

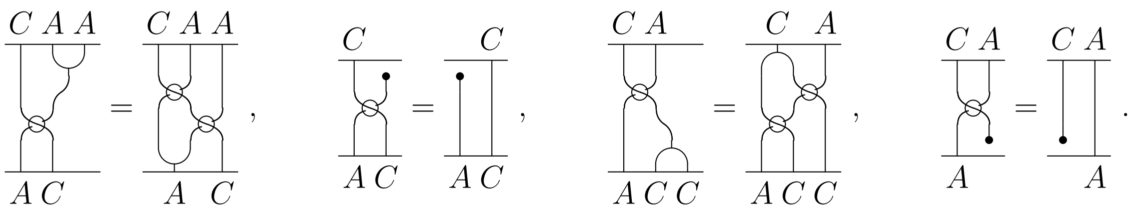
1. Basic definitions and tools

Let $\mathcal{M} = (\mathcal{M}, \otimes, I, \alpha, \lambda, \mu, \sigma)$ be a symmetric monoidal category. The main application we envision is $\mathcal{M} = \mathbb{K}$-Mod, the category of $\mathbb{K}$-modules over a commutative ring $\mathbb{K}$.

Let A be an algebra and C a coalgebra in $\mathcal{M}$. An *entwining* is a linear map $\psi : C \otimes A \longrightarrow A \otimes C$ such that

$$\psi(1_C \otimes \nabla_A) = (\nabla_A \otimes 1_C)(1_A \otimes \psi)(\psi \otimes 1_A),$$
$$\psi(1_C \otimes \eta_A) = \eta_A \otimes 1_C,$$
$$(1_A \otimes \Delta_C)\psi = (\psi \otimes 1_C)(1_C \otimes \psi)(\Delta_C \otimes 1_A),$$
$$(1_A \otimes \varepsilon_C)\psi = \varepsilon_C \otimes 1_A,$$

or in terms of braid diagrams



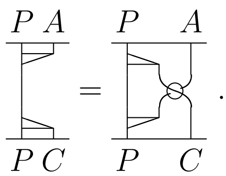
(1)

The triple (A, C, ψ) is called an *entwined structure*.

Let P in $\mathcal{M}$ be a right A-module (with structure map $\mu_P : P \otimes A \longrightarrow P$) and simultaneously a right C-comodule (with structure map $\delta_P : P \longrightarrow P \otimes C$). It is called an *entwined module* if

$$\delta_P \mu_P = (\mu_P \otimes 1_C)(1_P \otimes \psi)(\delta_P \otimes 1_A)$$

or in terms of a braid diagram



The conditions (??) for an entwining structure are necessary and sufficient for the structure map μ_P to be a comodule homomorphism and the structure map δ_P to be a module homomorphism [?]. A morphism of entwined modules is a morphism in $\mathcal{M}$ which is a homomorphism of A-modules and of C-comodules. The category of entwined modules is denoted by $\mathcal{M}_A^C(\psi)$.

A category $\mathcal{C}$ together with a functor $\otimes : \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$ and natural isomorphisms $\beta : (V \otimes W) \otimes P \longrightarrow V \otimes (W \otimes P)$ and $\zeta : I \otimes P \longrightarrow P$ is called an $\mathcal{M}$-*category* if the morphisms $\alpha, \lambda, \rho, \beta, \zeta$ are coherent in the obvious sense.

Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{M}$-categories. A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ together with a natural isomorphism $\xi : \mathcal{F}(V \otimes P) \longrightarrow V \otimes \mathcal{F}(P)$ is called an $\mathcal{M}$-*functor* if all associated morphisms are coherent.

Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be $\mathcal{M}$-functors. A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is called an $\mathcal{M}$-*transformation* if the diagrams

$$\begin{tikzcd}
\mathcal{F}(X \otimes P) \arrow[r, "\varphi(X \otimes P)"] \arrow[d, "\xi_{\mathcal{F}}"'] & \mathcal{G}(X \otimes P) \arrow[d, "\xi_G"] \\
X \otimes \mathcal{F}(P) \arrow[r, "X \otimes \varphi(P)"] & X \otimes \mathcal{G}(P)
\end{tikzcd}$$

commute.

Observe that $\mathcal{M}$ and $\mathcal{M}_A^C(\psi)$ are $\mathcal{M}$-categories and the underlying functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ is an $\mathcal{M}$-functor. If U and V are in $\mathcal{M}$ then $\omega \otimes U$ (with $(\omega \otimes U)(P) := \omega(P) \otimes U$) and $\omega \otimes V$ are $\mathcal{M}$-functors. We are interested in the $\mathcal{M}$-transformations $\varphi : \omega \otimes U \longrightarrow \omega \otimes V$.

Since $\mathcal{M}$ is symmetric we can similarly define an $\mathcal{M}^n$-functor in n-variables, in particular functors $\omega^n \otimes U$ given by $(\omega^n \otimes U)(P_1, \ldots, P_n) := \omega(P_1) \otimes \ldots \otimes \omega(P_n) \otimes U$. Furthermore we can define $\mathcal{M}^n$-transformations. The set of $\mathcal{M}^n$-transformations from $\omega^n \otimes U$ to $\omega^n \otimes V$ is denoted by $\mathrm{Nat}_{\mathcal{M}^n}(\omega^n \otimes U, \omega^n \otimes V)$. If $n = 0$, then $\mathcal{M}^0$ is the one point category and $\omega^0(\emptyset) \cong I$.

In [**?**, Theorem 3.1] we proved the following

Theorem 1.1. *Let (A, C, ψ) be an entwined structure. Then there is a natural (in U and V) isomorphism*

$$Nat_{\mathcal{M}^n}(\omega^n \otimes U, \omega^n \otimes V) \cong \mathcal{M}(C^{\otimes n} \otimes U, A^{\otimes n} \otimes V).$$

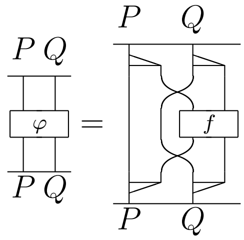
Proof. We gave the proof only for the functor ω^2, but it extends easily to functors of the form $\omega^n \otimes U$ as remarked in [**?**] after the proof of Theorem 2.1. We only give the map and its inverse without checking the details.

If $\varphi : \omega^n \otimes U \longrightarrow \omega^n \otimes V$ is an $\mathcal{M}^n$-transformation then it induces a morphism in $\mathcal{M}$

$$((1_A \otimes \varepsilon_C)^n \otimes 1_V)\varphi(A \otimes C, \ldots, A \otimes C)((\eta_A \otimes 1_C)^n \otimes 1_U) : C^{\otimes n} \otimes U \longrightarrow A^{\otimes n} \otimes V.$$

If $f : C^{\otimes n} \otimes U \longrightarrow A^{\otimes n} \otimes V$ is a morphism in $\mathcal{M}$ then $(\mu_{P_1} \otimes \ldots \otimes \mu_{P_n} \otimes 1_V)\pi(1_{P_1} \otimes \ldots \otimes 1_{P_n} \otimes f)(\delta_{P_1} \otimes \ldots \otimes \delta_{P_n} \otimes 1_U)$ is an $\mathcal{M}^n$-transformation (where π is a suitable permutation of the tensor factors in $\mathcal{M}$). These two maps are inverses of each other.

The last construction can be better viewed by the following braid diagram (for $n = 2$)



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It says that every $\mathcal{M}^n$-transformation φ is induced by a unique morphism f in the described way. $\square$

2. Morphisms of entwined structures and forgetful functors

Let (A, C, ψ) and (A', C', ψ') be entwined structures. In this section we want to study "forgetful" functors $\nu : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ such that the diagram

$$\begin{tikzcd} \mathcal{M}_A^C(\psi) \arrow[rr, "\nu"] \arrow[dr, "\omega"'] & & \mathcal{M}_{A'}^{C'}(\psi') \arrow[dl, "\omega'"] \\ & \mathcal{M} & \end{tikzcd} \tag{2}$$

commutes. We want to define morphisms between the entwined structures in such a way that the induced forgetful functor has this property.

Assume that such a functor ν satisfying (**??**) is given. For $P = (P, \mu_P, \delta_P) \in \mathcal{M}_A^C(\psi)$ we have $\nu(P) = (P, \mu'_P, \delta'_P) \in \mathcal{M}_{A'}^{C'}(\psi')$. In particular the A'-module structure on P induces a natural transformation $\mu' : \omega \otimes A' \longrightarrow \omega$. This transformation is an $\mathcal{M}$-transformation. So by Theorem **??** there is a unique morphism $f : C \otimes A' \longrightarrow A$ in $\mathcal{M}$ that induces μ'. Similarly there is a unique morphism $g : C \longrightarrow A \otimes C'$ in $\mathcal{M}$ that induces $\delta' : \omega \longrightarrow \omega \otimes C'$.

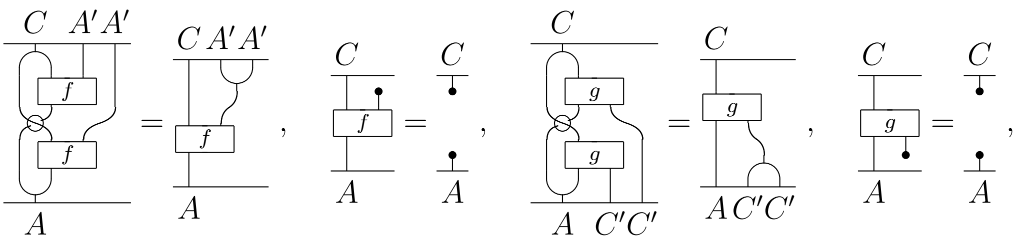
Definition 2.1. Let (A, C, ψ) and (A', C', ψ') be entwined structures. A pair of morphisms $(f : C \otimes A' \longrightarrow A, g : C \longrightarrow A \otimes C')$ is called a *morphism of entwined structures*

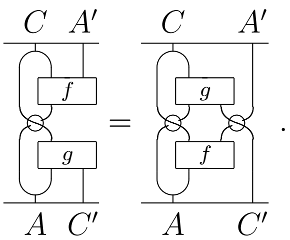
$$(f, g) : (A, C, \psi) \longrightarrow (A', C', \psi')$$

if the following hold:

(1) $\nabla_A(1_A \otimes f)(\psi \otimes 1_{A'})(1_C \otimes f \otimes 1_{A'})(\Delta_C \otimes 1_{A'} \otimes 1_{A'}) = f(1_C \otimes \nabla_{A'})$,
(2) $f(1_C \otimes \eta_{A'}) = \eta_A \varepsilon_C$,
(3) $(\nabla_A \otimes 1_{C'} \otimes 1_{C'})(1_A \otimes g \otimes 1_{C'})(\psi \otimes 1_{C'})(1_C \otimes g)\Delta_C = (1_A \otimes \Delta_{C'})g$,
(4) $(1_A \otimes \varepsilon_{C'})g = \eta_A \varepsilon_C$,
(5) $(\nabla_A \otimes 1_{C'})(1_A \otimes g)\psi(1_C \otimes f)(\Delta_C \otimes 1'_A) = (\nabla_A \otimes 1_{C'})(1_A \otimes f \otimes 1_{C'})(\psi \otimes \psi')(1_C \otimes g \otimes 1_{A'}))(\Delta_C \otimes 1'_A)$.

The associated braid diagrams (to be read from top to bottom) are

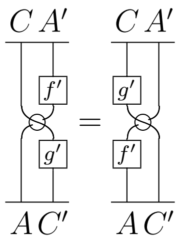




Remark 2.2. (1) In case of $f = \varepsilon_C \otimes f'$ and $g = \eta_A \otimes g'$ for morphisms $f' : A' \longrightarrow A$ and $g' : C \longrightarrow C'$ the conditions (1) - (5) reduce to the requirement that f' is a morphism of algebras, g' is a morphism of coalgebras and that

$$(1_A \otimes g')\psi(1_C \otimes f') = (f' \otimes 1_{C'})\psi'(g' \otimes 1_{A'})$$

or as braid diagram



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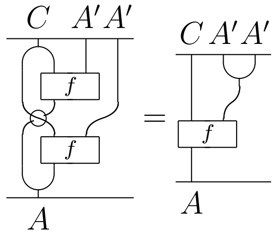
(2) The conditions (1) and (2) say essentially, that $f : C \otimes A' \longrightarrow A$ is an entwined measuring from A' to A by C. The conditions (3) and (4) say, that $g : C \longrightarrow A \otimes C'$ is an entwined comeasuring from C to C' by A. Condition (5) has to be considered as a compatibility condition for the measuring and the comeasuring.

Theorem 2.3. *Let (A, C, ψ) and (A', C', ψ') be entwined structures. There is a bijection between forgetful $\mathcal{M}$-functors $\nu : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ satisfying (2) and morphisms $(f, g) : (A, C, \psi) \longrightarrow (A', C', \psi')$.*

Proof. A functor $\nu : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ making the diagram (**??**) commutative is given by defining functorially the structure of an entwined module over (A', C', ψ') for each module $P \in \mathcal{M}_A^C(\psi)$. That means that we have to construct natural $\mathcal{M}$-transformations $\mu'_P : \omega(P) \otimes A' \longrightarrow \omega(P)$ and $\delta'_P : \omega(P) \longrightarrow \omega(P) \otimes C'$ that make each $P \in \mathcal{M}_A^C(\psi)$ into an entwined module over (A', C', ψ'). By Theorem **??** they are completely described by morphisms $f : C \otimes A' \longrightarrow A$ and $g : C \longrightarrow A \otimes C'$. We check now all the conditions for P to become an entwined module over (A', C', ψ') and give equivalent conditions in terms of f and g.

1) The associativity of the A' operation on P leads to two $\mathcal{M}$-transformations $\omega \otimes A' \otimes A' \longrightarrow \omega$ that are equal. So we get



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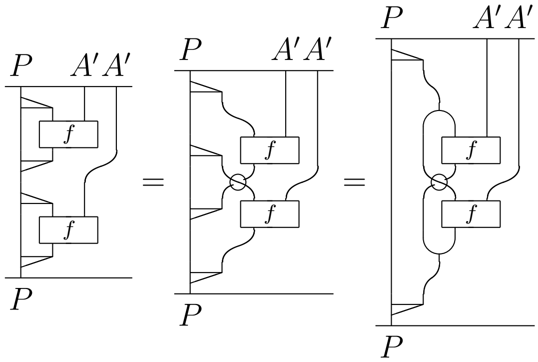
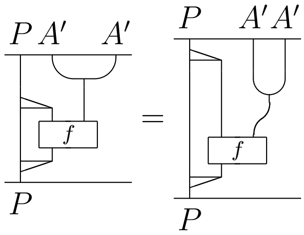
We will only show in this instance how to get to this equivalent condition in terms of braid diagrams. The following two $\mathcal{M}$-transformations are equal





iff the A'-operation on P is associative iff the two morphism that occur on the right hand side of these equations and define the $\mathcal{M}$-transformations are equal. This is equation (1) in the definition of morphisms of entwined structures.

2) With a similar argument one shows that the operation of A' on P is unary iff equation (2) holds in the definition of morphisms of entwined structures.

3) The cooperation of C' on P is coassociative iff equation (3) holds.

4) The cooperation of C' on P is counary iff equation (4) holds.

5) The A'-module and C'-comodule structure on P define an entwined module iff equation (5) holds. □

We remark that in the case $\mathcal{M} = \mathbb{K}$-Mod, the category of modules over a commutative ring $\mathbb{K}$, the requirements to be $\mathcal{M}$-functors resp. $\mathcal{M}$-transformations are automatically satisfied for any $\mathbb{K}$-additive functor and any natural transformation by [?] Proposition A.3 and Theorem A.4.

3. Tensor products on entwined modules

In this section we use again Theorem **??** and the same techniques as in the preceding section to study monoidal structures on the category of entwined modules that are defined on the underlying tensor product in $\mathcal{M}$.

Theorem 3.1. *Let (A, C, ψ) be an entwined structure. There is a bijection between $\mathcal{M}^2$-monoidal structures on $\mathcal{M}_A^C(\psi)$ preserved by the underling $\mathcal{M}$-functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ and quadruples of morphisms*

- $\widetilde{\Delta}_A : C \otimes C \otimes A \longrightarrow A \otimes A,$
- $\widetilde{\nabla}_C : C \otimes C \longrightarrow A \otimes A \otimes C,$
- $\widetilde{\varepsilon}_A : A \longrightarrow I,$

- $\widetilde{\eta}_C : I \longrightarrow C$,

satisfying the following identities

(1) $(\nabla_A \otimes \nabla_A)(1_A \otimes \sigma \otimes 1_A)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A)(1_A \otimes \sigma \otimes 1_C \otimes 1_A)(\psi \otimes \psi \otimes 1_A)(1_C \otimes \sigma \otimes 1_A \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\Delta}_A \otimes 1_A)(1_C \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_A \otimes 1_A)$
$= \widetilde{\Delta}_A(1_C \otimes 1_C \otimes \nabla_A)$,

(2) $\widetilde{\Delta}_A(1_C \otimes 1_C \otimes \eta_A) = (\eta_A \otimes \eta_A)(\varepsilon_C \otimes \varepsilon_C)$,

(3) $(\nabla_A \otimes \nabla_A \otimes 1_C \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\nabla}_C \otimes 1_C)(1_A \otimes \sigma \otimes 1_C \otimes 1_C)(\psi \otimes \psi \otimes 1_C)(1_C \otimes \sigma \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C)(1_C \otimes \sigma \otimes 1_C)(\Delta_C \otimes \Delta_C)$
$= (1_A \otimes 1_A \otimes \Delta_C)\widetilde{\nabla}_C$

(4) $(1_A \otimes 1_A \otimes \varepsilon_C)\widetilde{\nabla}_C = (\eta_A \otimes \eta_A)(\varepsilon_C \otimes \varepsilon_C)$,

(5) $(\nabla_A \otimes \nabla_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\nabla}_C)(1_A \otimes \sigma \otimes 1_C)(\psi \otimes \psi)(1_C \otimes \sigma \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\Delta}_A)(1_C \otimes \sigma \otimes 1_C \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_A)$
$= (\nabla_A \otimes \nabla_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_C)(\psi \otimes \psi \otimes \psi)(1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C)(1_C \otimes \sigma \otimes 1_C \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_A)$,

(6) $\widetilde{\varepsilon}_A \otimes \widetilde{\varepsilon}_A = \widetilde{\varepsilon}_A \nabla_A$,

(7) $\widetilde{\varepsilon}_A \eta_A = 1_I$,

(8) $\widetilde{\eta}_C \otimes \widetilde{\eta}_C = \Delta_C \widetilde{\eta}_C$,

(9) $\varepsilon_C \widetilde{\eta}_C = 1_I$,

(10) $(\widetilde{\varepsilon}_A \otimes 1_C)\psi(\widetilde{\eta}_C \otimes 1_A) = \widetilde{\eta}_C \widetilde{\varepsilon}_A$,

(11) $\varepsilon_C \otimes 1_A = (1_A \otimes \widetilde{\varepsilon}_A)\widetilde{\Delta}_A(1_C \otimes \widetilde{\eta}_C \otimes 1_A) = (\widetilde{\varepsilon}_A \otimes 1_A)\widetilde{\Delta}_A(\widetilde{\eta}_C \otimes 1_C \otimes 1_A)$,

(12) $\eta_A \otimes 1_C = (1_A \otimes \widetilde{\varepsilon}_A \otimes 1_C)\widetilde{\nabla}_C(1_C \otimes \widetilde{\eta}_C) = (\widetilde{\varepsilon}_A \otimes 1_A \otimes 1_C)\widetilde{\nabla}_C(\widetilde{\eta}_C \otimes 1_C)$,

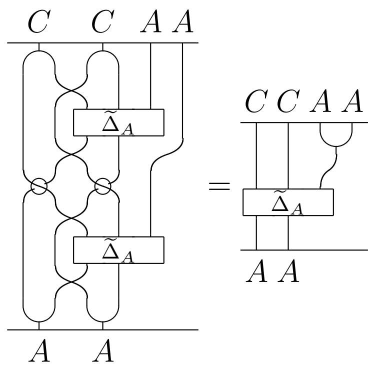
(13) $(\nabla_A \otimes \nabla_A \otimes 1_A)(1_A \otimes \sigma \otimes 1_A \otimes 1_A)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_A)(1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_A)(\psi \otimes \psi \otimes \widetilde{\Delta}_A)(1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C \otimes 1_C \otimes 1_A)(1_C \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_C \otimes 1_A)$
$= (1_A \otimes \nabla_A \otimes \nabla_A)(\sigma \otimes \sigma \otimes 1_A)(1_A \otimes \sigma \otimes \widetilde{\Delta}_A)(1_A \otimes 1_A \otimes \sigma \otimes 1_A)(1_A \otimes 1_A \otimes 1_A \otimes \sigma \otimes 1_A)(1_A \otimes 1_A \otimes 1_A \otimes 1_A \otimes \widetilde{\Delta}_A)(1_A \otimes 1_A \otimes 1_A \otimes \sigma \otimes 1_C \otimes 1_A)(1_A \otimes 1_A \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_A)(1_A \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_C \otimes 1_A)(\sigma \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_A)(1_C \otimes \psi \otimes \psi \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes 1_C \otimes \widetilde{\nabla}_C \otimes 1_A)(1_C \otimes \Delta_C \otimes \Delta_C \otimes 1_A)$,

(14) $(\nabla_A \otimes \nabla_A \otimes 1_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_A \otimes 1_C)(\psi \otimes \psi \otimes \widetilde{\nabla}_C)(1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_C)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C \otimes 1_C)(1_C \otimes \sigma \otimes 1_C \otimes 1_C)(\Delta_C \otimes \Delta_C \otimes 1_C)$
$= (1_A \otimes \nabla_A \otimes \nabla_A \otimes 1_C)(1_A \otimes 1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_C)(1_A \otimes 1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_C)(1_A \otimes \psi \otimes \psi \otimes 1_A \otimes 1_C)(\sigma \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_C)(1_C \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes 1_C \otimes \sigma \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes 1_C \otimes 1_C \otimes \widetilde{\nabla}_C)(1_C \otimes 1_C \otimes 1_C \otimes \sigma \otimes 1_C)(1_C \otimes 1_C \otimes \sigma \otimes 1_C)(1_C \otimes \sigma \otimes \widetilde{\nabla}_C)(\sigma \otimes \sigma \otimes 1_C)(1_C \otimes \Delta_C \otimes \Delta_C)$.

Proof. As in the proof of Theorem **??** we determine the conditions in terms of braid diagrams which can then be translated into the given formulas.

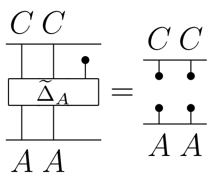
By Theorem **??** we have $\mathrm{Nat}_{\mathcal{M}^2}(\omega \otimes \omega \otimes A, \omega \otimes \omega) \cong \mathcal{M}(C \otimes C \otimes A, A \otimes A)$. So the tensor product of two modules P and Q in $\mathcal{M}_A^C(\psi)$ gets an A-structure by a uniquely determined morphism $\widetilde{\Delta}_A : C \otimes C \otimes A \longrightarrow A \otimes A$.

The A-operation on $P \otimes Q$ is associative iff



(1)

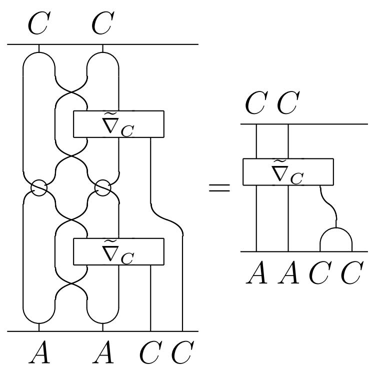
The A-operation on $P \otimes Q$ is unary iff



(2)

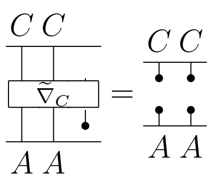
By Theorem **??** we also have $\mathrm{Nat}_{\mathcal{M}^2}(\omega \otimes \omega, \omega \otimes \omega \otimes C) \cong \mathcal{M}(C \otimes C, A \otimes A \otimes C)$. So the tensor product of two modules P and Q in $\mathcal{M}_A^C(\psi)$ gets a C-comodule structure by a morphism $\widetilde{\nabla}_C : C \otimes C \longrightarrow A \otimes A \otimes C$.

The C-cooperation on $P \otimes Q$ is coassociative iff



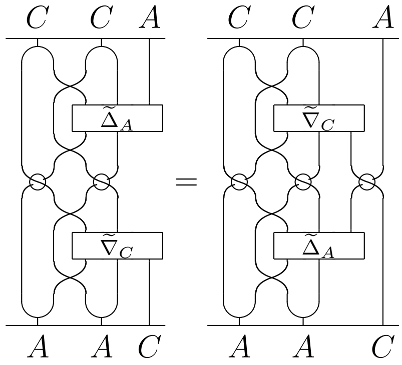
(3)

The C-cooperation on $P \otimes Q$ is counary iff



(4)

The tensor product $P \otimes Q$ is entwined with the given A-module and C-comodule structures iff

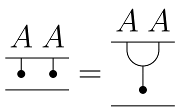


(5)

So equations (1)-(5) describe the tensor functor $\otimes : \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_A^C(\psi)$.

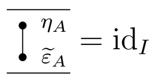
Now we study the special structures on the neutral object I. By Theorem **??** we have $\mathrm{Nat}(\omega^0 \otimes A, \omega^0) \cong \mathcal{M}(A, I)$. So the A-operation on I is described by a morphism $\widetilde{\varepsilon}_A : A \longrightarrow I$.

The A-operation on I is associative iff



(6)

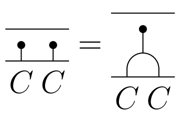
The A-operation on I is unary iff



(7)

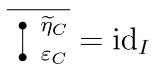
We also have $\mathrm{Nat}(\omega^0, \omega^0 \otimes C) \cong \mathcal{M}(I, C)$. So the C-cooperation on I is described by a morphism $\widetilde{\eta}_C : I \longrightarrow C$.

The C-cooperation on I is coassociative iff



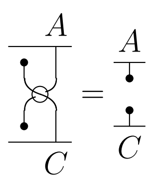
(8)

The C-cooperation on I is counary iff



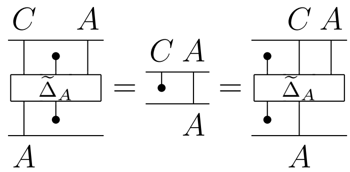
(9)

The object I is entwined with the given A-module and C-comodule structures iff



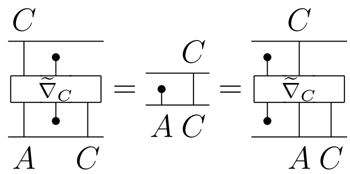
(10)

The object I acts as a left and right neutral object in $\mathcal{M}_A^C(\psi)$, i.e. λ and ρ in $\mathcal{M}$ are morphisms (for the A-module structure and the C-comodule structure) of entwined modules iff



(11)

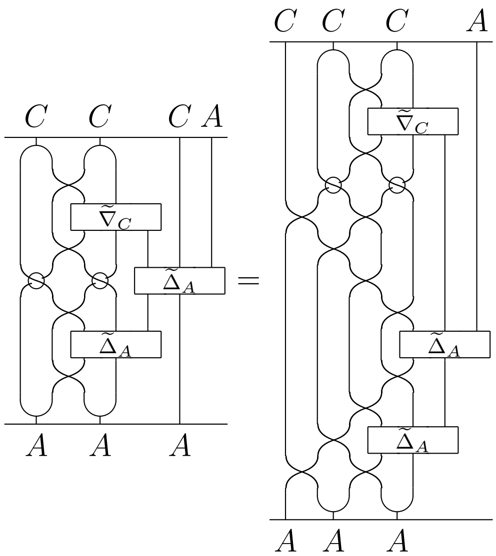
and



(12)

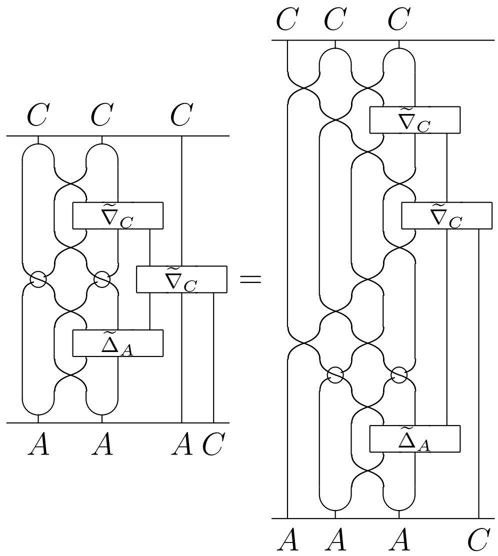
where the first equations comes from $\rho : \omega \otimes I \cong \omega$ and the second equations from $\lambda^{-1} : \omega \cong I \otimes \omega$.

Finally we study the associativity morphism $\alpha : (P \otimes Q) \otimes R \longrightarrow P \otimes (Q \otimes R)$. It is a morphism of A-modules iff



(13)

α is a morphism of C-comodules iff



(14)

□

Corollary 3.2. *Let (A, C, ψ) be an entwined structure. An $\mathcal{M}^2$-monoidal structure on $\mathcal{M}_A^C(\psi)$ preserved by the underlying $\mathcal{M}$-functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ is given by morphisms of the form*

- - $\widetilde{\Delta}_A = \varepsilon_C \otimes \varepsilon_C \otimes \Delta_A$ *with* $\Delta_A : A \longrightarrow A \otimes A$,
- - $\widetilde{\nabla}_C = \eta_A \otimes \eta_A \otimes \nabla_C$ *with* $\nabla_C : C \otimes C \longrightarrow C$,
- - $\widetilde{\varepsilon}_A = \varepsilon_A : A \longrightarrow I$,
- - $\widetilde{\eta}_C = \eta_C : I \longrightarrow C$,

iff the following identities hold:

(1) $(\nabla_A \otimes \nabla_A)(1_A \otimes \sigma \otimes 1_A)(\Delta_A \otimes \Delta_A) = \Delta_A \nabla_A$,
(2) $\Delta_A \eta_A = \eta_A \otimes \eta_A$,
(3) $(\nabla_C \otimes \nabla_C)(1_C \otimes \sigma \otimes 1_C)(\Delta_C \otimes \Delta_C) = \Delta_C \nabla_C$,
(4) $\varepsilon_C \nabla_C = \varepsilon_C \otimes \varepsilon_C$,
(5) $(1_A \otimes 1_A \otimes \nabla_C)(1_A \otimes \sigma \otimes 1_C)(\psi \otimes \psi)(1_C \otimes \sigma \otimes 1_A)(1_C \otimes 1_C \otimes \Delta_A) = (\Delta_A \otimes 1_C)\psi(\nabla_C \otimes 1_A)$,
(6) $\varepsilon_A \otimes \varepsilon_A = \varepsilon_A \nabla_A$,
(7) $\varepsilon_A \eta_A = \mathrm{id}_I$,
(8) $\eta_C \otimes \eta_C = \Delta_C \eta_C$,
(9) $\varepsilon_C \eta_C = \mathrm{id}_I$,
(10) $(\varepsilon_A \otimes 1_C)\psi(\eta_C \otimes 1_A) = \eta_C \varepsilon_A$,
(11) $(1_A \otimes \varepsilon_A)\Delta_A = 1_A = (\varepsilon_A \otimes 1_A)\Delta_A$,
(12) $\nabla_C(1_C \otimes \eta_C) = 1_C = \nabla_C(\eta_C \otimes 1_C)$,
(13) $(\Delta_A \otimes 1_A)\Delta_A = (1_A \otimes \Delta_A)\Delta_A$,
(14) $\nabla_C(\nabla_C \otimes 1_C) = \nabla_C(1_C \otimes \nabla_C)$.

Remark 3.3. These are essentially the conditions worked out in [?, Theorem 4.1] where we assumed that A and C are bialgebras in the first place without any additional (co-)action upon each other. The conditions arise from Theorem **??** by substituting the particular form of the morphisms $\widetilde{\Delta}_A$ and $\widetilde{\nabla}_C$ and using the properties of an entwined structure. So in this situation we get from the monoidal structure that A and C are bialgebras (conditions (1) – (4), (6) – (9), and (11) – (14)). The conditions (5) and (10) describe the compatibility with the given entwining and are exactly the conditions of [**?**, Theorem 4.1].

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