Compact extensions and contigual supernearness

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This paper is dedicated to our friend and colleague Dieter Pumplün on the occasion of his 70th birthday.

Abstract

Each Efremovic-proximity space (X, δ) has a compact Hausdorff extension Y so that two sets in X are near iff their closures in Y have a non-empty intersection. Moreover, X can be viewed as a dense subspace of Y. Lodato and Doítchinov generalized these results by considering more genral proximity structures such as Lodato-proximities and certain supertopologies, and by droping the Hausdorff requirement for Y. Here we study so-called supernearness spaces, a common generalization of Herrlich's nearness spaces and supertopological spaces, and show that each "contigual" supernearness space admits a compact topological extension as described above.

1 Introduction

Topological extensions are closely related to near-structures of various kinds. As a *classical* example we mention the **Smirnov compactification** [19] of a proximity space X that is a compact Hausdorff space Y, which contains X as a dense subspace and for which it is true that a pair of subsets of X is near iff their closures in Y meet. **Lodato** [16] [17] generalized this result to weaker conditions for the proximity and the space Y using "bunches" for the characterization of the extension. Ivanova and Ivanov [10] studied contiguity spaces and bicompact extensions such that a finite family of subsets of X are contigual iff there is a point of Y that is simultaneously in the closure in Y of each set in the family.

Herrlich [8] found a useful generalization of contiguity spaces by introducing nearness spaces, and Bentley [2] showed that bunch-determined nearness spaces are closely related to certain topological extensions.

Doítchinov [5] introduced the notion of supertopological spaces in order to construct a **unified** theory of topological, proximity and uniform spaces, and he proved a certain relationship of some special classes of supertopologies – called b-supertopologies – with compactly determined extensions.

Recently, supernear spaces were introduced by the author [12] [13] [14] in order to define a common generalization of nearness spaces and supertopological spaces as well. A special class of the so-called "clump-determined" supernear spaces are in one-to-one correspondence with certain symmetrical extensions and, moreover, in the non-symmetrical case we also have a neat internal characterization of the corresponding supernear spaces.

In this paper we study the relationship between compact topological extensions and the so-called "contigual supernear spaces", which are a common generalization of the supertopological spaces as well as the Lodato-proximity spaces.

2 Supernear spaces

As usual, $\boldsymbol{P}X$ denotes the power set of X, and we use $\mathscr{B}^X$ to denote a collection of *bounded* subsets of X, also known as a $\boldsymbol{B}$-*set*, *i.e.*, $\mathscr{B}^X \subseteq \boldsymbol{P}X$ satisfies the following three axioms:

(B1) $B' \subseteq B \in \mathscr{B}^X$ implies $B' \in \mathscr{B}^X$;

(B2) $\emptyset \in \mathscr{B}^X$;

(B3) $x \in X$ implies $\{x\} \in \mathscr{B}^X$.

If $\mathscr{B}^X$ and $\mathscr{B}^Y$ are $\boldsymbol{B}$-sets on X and Y, respectively, a function $f : X \to Y$ is called *bounded*, if it preserved bounded sets.

We recall the *corefinement* relation $\ll$ on $\boldsymbol{P}(\boldsymbol{P}X)$ given by $\mathscr{S}_2 \ll \mathscr{S}_1 :\Longleftrightarrow \forall F_2 \in \mathscr{S}_2 \, \exists F_1 \in \mathscr{S}_1 . \, F_2 \supseteq F_1$. For brevity we also write $\mathscr{S}_2 \cup \mathscr{S}_1$ for the set $\{ F_1 \cup F_2 \mid F_1 \in \mathscr{S}_1, F_2 \in \mathscr{S}_2 \}$.

2.1 Definition. For a $\boldsymbol{B}$-set $\mathscr{B}^X$ a function $S : \mathscr{B}^X \to \boldsymbol{P}(\boldsymbol{P}(\boldsymbol{P}X))$ is called a **supernear operator** or a **supernearness** on $\mathscr{B}^X$, and the pair $(\mathscr{B}^X, S)$ is called a **supernear(ness) space**, iff

(SN1) $B \in \mathscr{B}^X$ and $\mathscr{S}_2 \ll \mathscr{S}_1 \in S(B)$ imply $\mathscr{S}_2 \in S(B)$;

(SN2) $S(\emptyset) = \{\emptyset\}$ and $\mathscr{B}^X \notin S(B)$ for each $B \in \mathscr{B}^X$;

(SN3) $B' \subseteq B \in \mathscr{B}^X$ implies $S(B') \subseteq S(B)$;

(SN4) $x \in X$ implies $\{\{x\}\} \in S(\{x\})$;

(SN5) $B \in \mathscr{B}^X$ and $\mathscr{S}_1 \cup \mathscr{S}_2 \in S(B)$ imply $\mathscr{S}_1 \in S(B)$ or $\mathscr{S}_2 \in S(B)$;

(SN6) $B \in \mathscr{B}^X$ and $\{ cl_S(F) \mid F \in \mathscr{H} \} \in S(B)$ for some $\mathscr{S} \subseteq \boldsymbol{P}(\boldsymbol{P}X)$ imply $\mathscr{S} \in S(B)$, where $cl_S(F) := \{ x \in X \mid \{\{x\}, F\} \in S(\{x\}) \}$.

Elements of $N(B)$ are called **$\boldsymbol{B}$-near collections**. Given a pair of supernear spaces $(\mathscr{B}^X, S)$, $(\mathscr{B}^Y, T)$, a bounded map $f : \mathscr{B}^X \to \mathscr{B}^Y$ is called a **supernear map** or shortly **sn-map**, iff

(sn) $B \in \mathscr{B}^X$ and $\mathscr{S} \in S(B)$ imply $\{ f[F] \mid f \in \mathscr{S} \} \in T(f[B])$.

A map will also be referred to as a **supernear** map by saying it preserves $\boldsymbol{B}$-near collections in the above sense. We denote by $\boldsymbol{SN}$ the corresponding category.

2.2 Examples. Consider a $\boldsymbol{B}$-set $\mathscr{B}^X$ on X.

(i) For a nearness structure ζ on X we obtain a supernear operator on $\mathscr{B}^X$ by setting

$$S_\zeta(B) := \begin{cases} \{\emptyset\} & \text{if } B = \emptyset \\ \{ \mathscr{S} \subseteq \boldsymbol{P}X \mid \mathscr{S} \cup \{B\} \in \zeta \} & \text{otherwise} \end{cases}$$

(ii) For a Kuratowski closure operator *cl* on X, we obtain a supernear operator on $\mathscr{B}^X$ by setting

$$S_{\mathrm{cl}}(B) := \{ \mathscr{S} \subseteq \boldsymbol{P}X \mid B \in sec\{ cl(F) \mid F \in \mathscr{S} \} \}$$

where in general the operator *sec* on $\boldsymbol{P}(\boldsymbol{P}X)$ is defined by

$$sec\,\mathscr{M} := \{ T \subseteq X \mid \forall M \in \mathscr{M} . \, T \cap M \neq \emptyset \}$$

(iii) For a Leader-proximity [11] δ on X we obtain a supernear operator on $\mathscr{B}^X$ by setting

$$S_\delta(B) := \{ \mathscr{S} \subseteq \boldsymbol{P}X \mid \mathscr{S} \subseteq \delta(B) \}$$

where $\delta(B) := \{ F \subseteq X \mid B \delta F \}$.

(iv) For a quasi-uniformity $\mathscr{U}$ on X we obtain a supernear operator on $\mathscr{B}^X$ by setting

$$S_{\mathscr{U}}(B) := \left\{ \mathscr{S} \subseteq \boldsymbol{P}X \mid \forall U \in \mathscr{U}.\ \bigcap \{ U(F) \mid F \in \mathscr{S} \cup \{B\} \} \neq \emptyset \right\}$$

where $U(F) := \{ y \in X \mid \exists x \in F.\ (x, y) \in U \}$.

(v) For a supertopology θ on X (see [4]) we obtain a supernear operator on $\mathscr{B}^X$ by setting

$$S_\theta(B) := \{ \mathscr{S} \subseteq \boldsymbol{P}X \mid \mathscr{S} \subseteq sec\,\theta(B) \}$$

where $\theta(B)$ denotes the neighborhood system of B with respect to θ.

(vi) We first introduce the category $\boldsymbol{CEXT}$, whose objects are triples $E := (e, \mathscr{B}^X, Y)$ – called **compactly determined extensions** – where $X = (X, cl_X)$, $Y = (Y, cl_Y)$ are topological spaces (given by closure operators), $\mathscr{B}^X$ is a $\boldsymbol{B}$-set on X and $e : X \to Y$ is a function satisfying the following conditions:

(CE1) $A \in \boldsymbol{P}X$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$;

(CE2) $cl_Y(e[X]) = Y$, which means that the image of X under e is **dense** in Y.

(CE3) $x \in X$ and $y \in cl_Y(\{e(x)\})$ imply $e(x) \in cl_Y(\{y\})$, which means that Y is **symmetric relative** to $e[X]$.

(CE4) $\{ cl_Y(e[A]) \mid A \subseteq X \}$ is a **base** for the closed subsets of Y, which means that the extension E is **strict** in the sense of Banaschewski [1].

(CE5) For any $y \in Y$ there exists a set $A \subseteq X$ such that $y \in cl_Y(e[A])$, and $cl_Y(e[A])$ is compact, which means that the extension is compactly generated.

Morphisms in $\boldsymbol{CEXT}$ have the form $(f, g) : (e, \mathscr{B}^X, Y) \to (e', \mathscr{B}^{X'}, Y')$, where $f : X \to X'$, $g : Y \to Y'$ are **continuous** maps such that f is also **bounded**, and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[e']{} & Y' \end{array}$$

If $(f, g) : (e, \mathscr{B}^X, Y) \to (e', \mathscr{B}^{X'}, Y')$ and $(f', g') : (e', \mathscr{B}^{X'}, Y') \to (e'', \mathscr{B}^{X''}, Y'')$ are $\boldsymbol{CEXT}$-morphisms, then they can be **composed** according to the rule $(f', g') \circ (f, g) := (f' \circ f, g' \circ g) : (e, \mathscr{B}^X, Y) \to (e'', \mathscr{B}^{X''}, Y'')$, where "$\circ$" denotes the **composition** of maps.

Given a compactly determined extension $E = (e, \mathscr{B}^X, Y)$, we now obtain a supernear operator on $\mathscr{B}^X$ by setting

$$S^E(B) := \{ \mathscr{S} \subseteq \boldsymbol{P}X \mid \forall F \in \mathscr{S} \exists y \in cl_Y(e[B]).\ y \in cl_Y(e[F]) \}$$

2.3 Remark. We pointed out that – in correspondence to the above-mentioned examples – the category $\boldsymbol{SN}$ of supernear spaces contains the following categories as full subcategories:

- the category $\boldsymbol{TOP}$ of topological spaces and continuous maps;

- the category $\boldsymbol{PROX}_{\mathrm{Le}}$ of Leader proximity spaces and δ-maps, hence also $\boldsymbol{PROX}_{\mathrm{Lo}}$, the category whose objects are Lodato proximity spaces;

- the category $\boldsymbol{NEAR}$ of nearness spaces and nearness-preserving maps;

- the category $\boldsymbol{CONT}$ of contiguity spaces and c-maps;
- the category $\boldsymbol{UNIF}$ of uniform spaces and uniformly continuous maps; and at last
- the category $\boldsymbol{STOP}$ of supertopological spaces and bounded continuous maps.

2.4 Lemma. *For a compactly determined extension $E = (e, \mathscr{B}^X, Y)$ the supernear operator S^E of Example 2.2(vi) has the following additional properties:*

(S) S^E *is* **symmetric**, *which means*

$$B \in \mathscr{B}^X \quad \text{and} \quad \mathscr{S} \in S^E(B) \quad \text{imply} \quad \{B\} \cup \mathscr{S} \in \bigcap \{ S^E(F) \mid F \in (\mathscr{S} \cap \mathscr{B}^X) \cup \{B\} \}$$

(A) S^E *is* **additive**, *which means*

$$B_1 \cup B_2 \in \mathscr{B}^X \quad \text{implies} \quad S^E(B_1 \cup B_2) \subseteq S^E(B_1) \cup S^E(B_2)$$

(CI) S^E *is* **closure-isotone**, *which means*

$$cl_{S^E}(B) \in \mathscr{B}^X \quad \text{implies} \quad S^E(cl_{S^E}(B)) \subseteq S^E(B)$$

(E) S^E *is* **endogenous**, *which means*

$$B \in \mathscr{B}^X \quad \text{implies} \quad \bigcup \{ \mathscr{S} \subseteq \boldsymbol{P}X \mid \mathscr{S} \in S^E(B) \} \in S^E(B)$$

Moreover, the closure operator cl_{S^E} coincides with the topological closure operator cl_X.

Proof: First we note that for each supernearness S on $\mathscr{B}^X$ the corresponding hull operator cl_S is always topological, in particular this applies to S^E. Then it is straightforward to verify the listed properties. In order to prove the equality of the closure operators, consider $A \in \boldsymbol{P}X$ and $x \in cl_X(A)$. Then, by (CE1), $e(x) \in cl_Y(e[A]) \cap cl_Y(\{e(x)\})$, hence $\{\{x\}, A\} \in S^E(\{x\})$. Thus $x \in cl_{S^E}(A)$.

Conversely, consider $x \in cl_{S^E}(A)$. Then $\{\{x\}, A\} \in S^E(\{x\})$, which implies $y \in cl_Y(e[A])$ for some $y \in cl_Y(\{e(x)\})$. As a consequence of (CE3) we get $e(x) \in cl_Y(cl_Y(e[A])) = cl_Y(e[A])$, hence in view of (CE1) we obtain $x \in e^{-1}[cl_Y(e[A])] = cl_X(A)$, which was to be shown. □

3 Functorial relationships between $\boldsymbol{CEXT}$ and $\boldsymbol{SN}$

Now, we are going to construct a functor from the category $\boldsymbol{CEXT}$ to the category $\boldsymbol{SN}$.

3.1 Theorem. *We obtain a functor $F : \boldsymbol{CEXT} \to \boldsymbol{SN}$ by setting*

(a) $F(E) := (\mathscr{B}^X, S^E)$; *for a compactly determined extension $E := (e, \mathscr{B}^X, Y)$*

(b) $F(f, g) := f$ *for a $\boldsymbol{CEXT}$-morphism $(f, g) : E := (e, \mathscr{B}^X, Y) \to E' := (e', \mathscr{B}^{X'}, Y')$*

Proof: In view of Lemma 2.4 we already know that $F(E)$ is an object of $\boldsymbol{SN}$ with the corresponding additional properties.

Now let $E := (e, \mathscr{B}^X, Y) : (f, g) \to E' := (e', \mathscr{B}^{X'}, Y')$ be a $\boldsymbol{CEXT}$-morphism. It has to be shown that f preserves the near-collections from $F(E) := (\mathscr{B}^X, S^E)$ to $F(E') := (\mathscr{B}^{X'}, S^{E'})$. Without loss of generality, let $B \in \mathscr{B}^X \setminus \{\emptyset\}$ and $\mathscr{S} \in S^E(B)$. Now consider $F \in \mathscr{S}$. By definition, there exists $y \in cl_Y(e[B])$ such that $y \in cl_Y(e[F])$. The hypothesis implies $g(y) \in g[cl_Y(e[B])]$ and therefore $g(y) \in cl_{Y'}(g[e[B]]) = cl_{Y'}(e'[f[B]])$, since (f, g) is a $\boldsymbol{CEXT}$-morphism. Because $y \in cl_Y(e[F])$, we have $g(y) \in cl_{Y'}(e'[f[F]])$, which results in $\{ f[F] \mid F \in \mathscr{S} \} \in S^{E'}(f[B])$. □

To obtain a related functor in the opposite direction, we introduce the notion of so-called B-clips for each bounded set $B \in \mathscr{B}^X \setminus \{\emptyset\}$. This is motivated by the following facts.

Given a (compactly determined) extension $E = (e, \mathscr{B}^X, Y)$, it is possible to define a function $t : Y \twoheadrightarrow \boldsymbol{P}(\boldsymbol{P}X)$ by setting

$$t(y) := \{ T \subseteq X \mid y \in cl_Y(e[T]) \}$$

Moreover, for each $B \in \mathscr{B}^X \setminus \{\emptyset\}$ we put

$$\mathscr{C}^B := \bigcup \{ t(y) \mid y \in cl_Y(e[B]) \}$$

Now every B-near collection $\mathscr{S} \in S^E(B)$ satisfies $\mathscr{S} \subseteq \mathscr{C}^B$; in fact $F \in \mathscr{S}$ implies the existence of some $y \in cl_Y(e[B])$ such that $y \in cl_Y(e[F])$, hence $F \in t(y)$ and consequently $F \in \mathscr{C}^B$.

This leads to the following definition.

3.2 Definition. Let $(\mathscr{B}^X, S)$ be a supernear space. For $B \in \mathscr{B}^X \setminus \{\emptyset\}$ a subset $\mathscr{C} \subseteq \boldsymbol{P}X$ is called a **B-clip** in S, provided that

(C1) $\emptyset \notin \mathscr{C}$;

(C2) $C_1 \in \mathscr{C}$ and $C_1 \subseteq C_2 \in \boldsymbol{P}X$ imply $C_2 \in \mathscr{C}$;

(C3) $C_1 \cup C_2 \in \mathscr{C}$ implies $C_1 \in \mathscr{C}$ or $C_2 \in \mathscr{C}$;

(C4) $B \in \mathscr{C}$;

(C5) $cl_S(C) \in \mathscr{C}$ implies $C \in \mathscr{C}$;

(C6) $\mathscr{C} \in S(B)$;

(C7) $\bigcap \{ cl_S(T) \mid T \in \mathscr{C} \} = \emptyset$ implies the existence of a finite subset $\mathscr{C}_0 \subseteq \mathscr{C}$ with $\bigcap \{ cl_S(T) \mid T \in \mathscr{C}_0 \} = \emptyset$

Another interesting example for this notion is given by the set system

$$e_X(x) := \{ T \subseteq X \mid x \in cl_S(T) \}$$

for $x \in X$, which is a $\{x\}$-clip in S. Moreover, $e_X(x)$ is a maximal element in $S(\{x\})$ ordered by set-inclusion. This can be shown as follows. Let $\mathscr{C}$ be an element of $S(\{x\})$ and assume $e_X(x) \subseteq \mathscr{C}$. By hypothesis we have $\{x\} \in \mathscr{C}$. Now, $C \in \mathscr{C}$ implies $\{\{x\}, C\} \in S(\{x\})$, because of $\{\{x\}, C\} \ll \mathscr{C}$. Hence we get $x \in cl_S(C)$ which means $C \in e_X(x)$.

With respect to the above-mentioned motivation and Remarks, we naturally arrive at the following definition.

3.3 Definition. A supernear space $(\mathscr{B}^X, S)$, as well as S, is called **clip-determined**, provided that

(CL) $B \in \mathscr{B}^X \setminus \{\emptyset\}$ and $\mathscr{S} \in S(B)$ imply the existence of a B-clip $\mathscr{C}$ with $\mathscr{S} \subseteq \mathscr{C}$.

3.4 Remark. In addition to the properties of Lemma 2.4, the supernearness S^E as defined in Example 2.2(vi) is also clip-determined.

We now prepare the introduction of a functor $G : \boldsymbol{SN} \twoheadrightarrow \boldsymbol{CEXT}$ in the opposite direction to F.

3.5 Lemma. *Let* $(\mathscr{B}^X, S)$ *be a supernear space. We put*

$$\hat{X} := \{\mathscr{C} \subseteq \boldsymbol{P}X \mid \exists B \in \mathscr{B}^X \setminus \{\emptyset\}. \mathscr{C} \text{ is a } B\text{-clip}\}$$

and for each $\hat{A} \subseteq \hat{X}$ *we set*

$$cl_{\hat{X}}(\hat{A}) := \{\, \mathscr{C} \in \hat{X} \mid \bigcap \hat{A} \subseteq \mathscr{C} \,\}$$

where $\bigcap \hat{A} := \{\, F \subseteq X \mid \forall \mathscr{C} \in \hat{A}.\ F \in \mathscr{C} \,\}$ *(so that, by convention,* $\bigcap \hat{A} = \boldsymbol{P}X$ *if* $\hat{A} = \emptyset$*). Then* $cl_{\hat{X}}$ *is a topological closure operator on* $\hat{X}$*.*

Proof: Straightforward. □

3.6 Theorem. *For supernear spaces* $(\mathscr{B}^X, S)$ *and* $(\mathscr{B}^Y, T)$ *let* $f : X \to Y$ *be an sn-map. Define a function* $\hat{f} : \hat{X} \to \hat{Y}$ *by setting for each* $\mathscr{C} \in \hat{X}$

$$\hat{f}(\mathscr{C}) := \{\, D \subseteq Y \mid f^{-1}[cl_T(D)] \in \mathscr{C} \,\}$$

Then the following statements are valid.

(1) $\hat{f}$ *is a continuous map from* $(\hat{X}, cl_{\hat{X}})$ *to* $(\hat{Y}, cl_{\hat{Y}})$*.*

(2) *The composites* $\hat{f} \circ e_X$ *and* $e_Y \circ \hat{f}$ *coincide, where* $e_X : X \to \hat{X}$ *is the function that assigns the* $\{x\}$*-clip* $e_X(x)$ *to* x*.*

(3) $\{\, f[C] \mid C \in \mathscr{C} \,\} \subseteq \hat{f}(\mathscr{C})$*.*

(4) $\bigcap e_X[B] := \bigcap \{\, e_X(x) \mid x \in B \,\} = \{\, F \subseteq X \mid B \in cl_S(F) \,\}$ *for every* $B \subseteq X$*.*

Proof: We prove statement (2), all other verifications are left to the reader. Let x be an element of X. We have to show the validity of $\hat{f}(e_X(x)) = e_Y(f(x))$. To this end, let $F \in e_Y(f(x))$. Then $f(x) \in cl_T(F)$, hence $x \in f^{-1}[cl_T(F)]$, and consequently $f^{-1}[cl_T(F)] \in e_X(x)$. Thus $F \in \hat{f}(e_X(x))$, which proves the inclusion $e_Y(f(x)) \subseteq \hat{f}(e_X(x))$. Since $e_Y(f(x))$ is maximal with respect to set-inclusion on $T(\{f(x)\}) \setminus \{\emptyset\}$ and since $\{\, cl_T(D) \mid D \in \hat{f}(e_X(x)) \,\}$ corefines $\{\, f[V] \mid V \in e_X(x) \,\}$, the hypothesis that f is an sn-map implies the desired equality. □

3.7 Remark. With respect to Lemma 2.4 and Remark 3.4 we summarize that the supernear operator S^E satisfies the axioms of being symmetric, additive, closure-isotone, endogenous and clip-determined.

These facts motivate the following notion.

3.8 Definition. A supernear operator on $\mathscr{B}^X$, and also the corresponding space, is called **contigual**, if the above-mentioned axioms for the operator are satisfied. Moreover, we denote the corresponding full subcategory of $\boldsymbol{SN}$ by $\boldsymbol{CSN}$.

3.9 Theorem. *We obtain a functor* $G : \boldsymbol{CSN} \to \boldsymbol{CEXT}$ *by setting*

(a) $G(\mathscr{B}^X, S) := (e_X, \mathscr{B}^X, \hat{X})$ *for any contigual supernear space* $(\mathscr{B}^X, S)$ *with* $X := (X, cl_S)$ *and* $\hat{X} := (\hat{X}, cl_{\hat{X}})$*;*

(b) $G(f) := (f, \hat{f})$ *for any sn-map* $f : (\mathscr{B}^X, S) \to (\mathscr{B}^Y, T)$*.*

Proof: In view of (SN6) it is straightforward to verify that cl_S is a topological closure operator on X. By Lemma 3.5, we also have the topological closure operator $cl_{\hat{X}}$ on $\hat{X}$. Therefore we obtain topological spaces with the $\boldsymbol{B}$-set $\mathscr{B}^X$, and $e_X : X \to \hat{X}$ is a continuous map according to Theorem3.6.

To establish (CE1), let A be a subset of X and suppose $x \in cl_S(A)$. Then, by Theorem 3.6(4) the inclusion $\bigcap e_X[A] \subseteq e_X(x)$ follows. This means that $e_X(x) \in cl_{\hat{X}}(e_X[A])$, hence $x \in e_X^{-1}[cl_{\hat{X}}(e[A])]$. Conversely, let x be an element of $\in e_X^{-1}[cl_{\hat{X}}(e[A])]$. Then by definition we have $e_X(x) \in cl_{\hat{X}}(e_X[A])$, and consequently $\bigcap e_X[A] \subseteq e_X(x)$. By Theorem 3.6(4) we obtain $A \in e_X(x)$, which means $x \in cl_S(A)$.

To establish (CE2), let $\mathscr{C} \in \hat{X}$ and suppose $\mathscr{C} \notin cl_{\hat{X}}(e_X[X])$. By definition we get $\bigcap e_X[X] \nsubseteq \mathscr{C}$, so that there exists a set $F \in \bigcap e_X[X]$ with $F \notin \mathscr{C}$. By Theorem 3.6(4) the inclusion $X \subseteq cl_X(F)$ holds. Since $B \in \mathscr{C}$ for some $B \in \mathscr{B}^X$ (see also (C2)) and in view of axiom (C4), we get $cl_S(F) \in \mathscr{C}$, hence $F \in \mathscr{C}$, because of axiom (C5). But this is a contradiction, which shows $\mathscr{C} \in cl_{\hat{X}}(e_X[X])$.

To establish (CE3), let x be an element of X such that $\mathscr{C} \in cl_{\hat{X}}(\{e(x)\})$. We must show $e_X(x) \in cl_{\hat{X}}(\{\mathscr{C}\})$. By hypothesis we have $e_X(x) \subseteq \mathscr{C}$ and moreover $\mathscr{C} \in S(B)$ for some $B \in \mathscr{B}^X \setminus \{\emptyset\}$. Since $\{x\} \in \mathscr{C}$ and since $\mathscr{C}$ is symmetric, we get $\{B\} \cup \mathscr{C} \in S(\{x\})$ with $\mathscr{C} \ll \{B\} \cup \mathscr{C}$. According to (SN1) we then get $\mathscr{C} \in S(\{x\})$, and since $e_X(x)$ is maximal with respect to $(S(\{x\}) \setminus \{\emptyset\}, \subseteq)$, $\mathscr{C}$ coincides with $e_X(x)$.

By hypothesis $f : (\mathscr{B}^X, S) \to (\mathscr{B}^Y, T)$ is an sn-map, in particular f is continuous and bounded. It remains to show that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{e_X} & \hat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ Y & \xrightarrow[e_Y]{} & \hat{Y} \end{array}$$

To this end let x be an element of X. We must show $(\hat{f} \circ e_X)(x) = (e_Y \circ f)(x)$.

"$\subseteq$": $D \in (\hat{f} \circ e_X)(x) = \hat{f}(e_X(x))$ means $f^{-1}[cl_T(D)] \in e_X(x)$, hence $x \in cl_S(f^{-1}[cl_T(D)])$. In particular we have $f(x) \in cl_T(f[f^{-1}[cl_T(D)]])$, since f is continuous. But now $cl_T(cl_T(D)) \subseteq cl_T(D)$ implies $D \in e_Y(f(x))$.

"$\supseteq$": $D \in e_Y(f(x))$ implies $f(x) \in cl_T(D)$, hence $x \in f^{-1}[cl_T(D)]$ and consequently $x \in f^{-1}[cl_T(D)])$. This implies $f^{-1}[cl_T(D)] \in e_X(x)$, which means $C \in \hat{f}(e_X(x))$. Finally, this establishes that the composition of sn-maps ist preserved by G.

Axiom (CE4) can be verified in an indirect manner, and (CE5) should be proven according to (C7) in the definition of a B-clip in S. $\square$

3.10 Theorem. *Let $F : \boldsymbol{CEXT} \to \boldsymbol{SN}$ and $G : \boldsymbol{CSN} \to \boldsymbol{CEXT}$ be the functors given in Theorems 3.1 and 3.9. For each object $(\mathscr{B}^X, S)$ of $\boldsymbol{CSN}$ let $t(\mathscr{B}^X, S)$ denote the identity map $t(\mathscr{B}^X, S) := id_X : F(G(\mathscr{B}^X, S)) \to (\mathscr{B}^X, S)$. Then $t : F \circ G \to 1_{\boldsymbol{CSN}}$ is a natural equivalence from $F \circ G$ to the identity functor $1_{\boldsymbol{CSN}}$, i.e., $id_X : F(G(\mathscr{B}^X, S)) \to (\mathscr{B}^X, S)$ is an isomorphism for each $\boldsymbol{CSN}$-object $(\mathscr{B}^X, S)$ and the following diagram commutes for each sn-map $f : (\mathscr{B}^X, S) \to (\mathscr{B}^Y, T)$*

$$\begin{array}{ccc} F(G(\mathscr{B}^X, S)) & \xrightarrow{id_X} & (\mathscr{B}^X, S) \\ {\scriptstyle F(G(f))}\downarrow & & \downarrow{\scriptstyle f} \\ F(G(\mathscr{B}^Y, T)) & \xrightarrow[id_Y]{} & (\mathscr{B}^Y, T) \end{array}$$

Proof: The commutativity of the diagram is obvious, since $F(G(f)) = f$. It remains to prove that $id_X : F(G(\mathscr{B}^X, S)) \to (\mathscr{B}^X, S)$ is an sn-map for each object $(\mathscr{B}^X, S)$ of $\boldsymbol{CSN}$ and vice versa. To fix the notation, let S' be such that $F(G(\mathscr{B}^X, S) = F(e_X, \mathscr{B}^X, \hat{X}) = (\mathscr{B}^X, S')$. It suffices to show that for each $B \in \mathscr{B}^X \setminus \{\emptyset\}$ we have $S'(B) \subseteq S(B)$. To this end assume $\mathscr{S}' \in S'(B)$. In view of Lemma 2.4(iv) it suffices to establish $\mathscr{S}' \subseteq \bigcup\{\, \mathscr{S} \subseteq \boldsymbol{P}X \mid \mathscr{S} \in S(B)\,\}$. But $F \in \mathscr{S}'$ implies the existence of an element $\mathscr{C} \in cl_{\hat{X}}(e[B])$ such that $\mathscr{C} \in cl_{\hat{X}}(e_X([F])$, hence $\bigcap e_X[B] \subseteq \mathscr{C}$.

In view of Theorem 3.6(4) we get $B \in \mathscr{C}$ and $\mathscr{C} \in S(B')$ for some $B' \in \mathscr{B}^X \setminus \{\emptyset\}$ (note in particular that $\mathscr{C}$ is a B'-clip for some bounded set B'). Since S is symmetric, we get $\{B'\} \cup \mathscr{C} \in S(B)$ and $\mathscr{C} \ll \{B'\} \cup \mathscr{C}$, hence $\mathscr{C} \in S(B)$ according to (SN1). On the other hand, we also know that the statement $\mathscr{C} \in cl_{\hat{X}}(e_X[F])$ holds, which implies $F \in \mathscr{C}$ according to Theorem 3.6(4) and the definition of the hull operator $cl_{\hat{X}}$, respectively.

In the opposite direction consider $\mathscr{S} \in S(B)$. Since S in particular is clip-determined, we can choose a B-clip $\mathscr{C}$ such that $\mathscr{S} \subseteq \mathscr{C}$. In order to show $\mathscr{S} \in S'(B)$ we need to verify that for $F \in \mathscr{S}$ we should have

(1) $\mathscr{C} \in cl_{\hat{X}}(e_X[B])$, and

(2) $\mathscr{C} \in cl_{\hat{X}}(e_X[F])$.

So let F be an element of $\mathscr{S}$.

(1) By definition of $cl_{\hat{X}}$ it suffices to establish $\bigcap e_X[B] \subseteq \mathscr{C}$. So let D be an element of $\bigcap e_X[B]$, which means $B \subseteq cl_S(D)$. Since $B \in \mathscr{C}$ according to (C4), we get $cl_S(D) \in \mathscr{C}$, hence $D \in \mathscr{C}$ by (C5).

(2) $D \in \bigcap e_X[F]$ implies $F \subseteq cl_S(D)$. Since $F \in \mathscr{C}$ by hypothesis, we get $cl_S(D) \in \mathscr{C}$, and analogously we infer $D \in \mathscr{C}$, which concludes the proof. □

Now we are able to formulate the main theorem of this paper, which is a consequence of the preceding Lemmata and Theorems, respectively.

3.11 Theorem. *Let $(\mathscr{B}^X, S)$ be a supernear space. Then the following are equivalent:*

(i) *$(\mathscr{B}^X, S)$ is contigual;*

(ii) *there exists a compact extension $(e, \mathscr{B}^X, Y)$ such that for each $B \in \mathscr{B}^X \setminus \{\emptyset\}$ the elements $\mathscr{S} \in S(B)$ are characterized by*

$$cl_Y(e[B]) \in sec\{\, cl_Y(e[F]) \mid F \in \mathscr{S}\,\}$$

(iii) *there exists a topological space (Y, cl_Y) and a continuous map $f : X \to Y$ that satisfies*

- *$cl_S(A) = f^{-1}[cl_Y(f[A])]$ for each $A \subseteq X$;*
- *$f[X]$ is dense in Y;*
- *Y is symmetric relative to $f[X]$;*
- *$\{\, cl_Y(e[A]) \mid A \subseteq X\,\}$ forms a base for the closed subsets of Y;*
- *$\forall y \in Y\, \exists A \subseteq X.\, y \in cl_Y(e[A])$ and $cl_Y(e[A])$ is compact;*
- *for each $\mathscr{S} \in \mathscr{B}^X \setminus \{\emptyset\}$ the elements $\mathscr{S} \in S(B)$ are characterized by the fact that for each $F \in \mathscr{S}$ there exists $y \in cl_Y(e[B])$ such that $y \in cl_Y(e[F])$.* □

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