The Graded Ring of Hermitian Modular Forms of Degree 2 over $\mathbb{Q}(\sqrt{-2})$

by

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Dedicated to Professor Dr. Hans-Dieter Pumplün
on the occasion of his 70th birthday

Abstract

We describe the graded ring of symmetric Hermitian modular forms of even weights and degree 2 over $\mathbb{Q}(\sqrt{-2})$ in terms of generators and relations. All the 8 generators of weight up to 12 are Maaß lifts and some of them can also be obtained from Borcherds products. Moreover we construct generators for the module over this ring consisting of all Hermitian modular forms with respect to the commutator subgroup. As an application the field of Hermitian modular functions over $\mathbb{Q}(\sqrt{-2})$ is determined. Finally we construct 5 algebraically independent symmetric Hermitian modular forms in terms of theta series.

1. Introduction

In the 1960's Igusa [Ig] described the graded ring of Siegel modular forms of degree 2. By the same method Freitag [F] was able to determine the graded ring of symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(\sqrt{-1})$, where Nagaoka and Ibukiyama completed a description in terms of generators and relations. Considering $\mathbb{Q}(\sqrt{-3})$ analogous results are known due to [DK]. Partial results on the case $\mathbb{Q}(\sqrt{-2})$ were obtained by Freitag and Hermann [FH] by embedding its ring of integers into the Hurwitz quaternions.
In this paper we describe the graded ring of symmetric Hermitian modular forms of even weights and degree 2 over $\mathbb{Q}(\sqrt{-2})$ completely. All the 8 generators are obtained as Maaß lifts or as Borcherds products similar to [DK]. We need the Siegel-Eisenstein series of weight $4, 6, 8, 10$ and 12, which are algebraically independent, as well as the three

products of two Hermitian modular forms with respect to the non-trivial abelian character (cf. Corollary 5). Unfortunately the method is much more involved than in [DK] since we cannot construct a Borcherds product with a simple divisor set. Any Borcherds product vanishes at least on two rational quadratic divisors of different discriminants. Moreover divisors of higher order occur (cf. Theorem 3). We have to construct liftings of Siegel modular forms with respect to the paramodular group of level 2 resp. 3 which were determined in [IO] resp. [D3]. As an application we can describe the graded ring with respect to the commutator subgroup and the field of Hermitian modular functions. Moreover we can construct 5 algebraically independent symmetric Hermitian modular forms in terms of theta series which already appear in the paper by Freitag and Hermann [FH].

2. The Maaß space

The *Hermitian half-space* of degree 2 is given by

$$H_2(\mathbb{C}) = \left\{ Z = \begin{pmatrix} \tau & z \\ w & \tau' \end{pmatrix} \in \mathbb{C}^{2\times 2}; \quad \frac{1}{2i}(Z - \overline{Z}^{tr}) > 0 \right\}$$

where *tr* stands for the transpose and τ, τ' belong to the upper half-plane $\mathcal{H}$ in $\mathbb{C}$. Let $K = \mathbb{Q}(i\sqrt{2})$ be the imaginary quadratic number field of discriminant -8 with its ring of integers

$$\mathcal{O} := \mathbb{Z} + \mathbb{Z}i\sqrt{2}$$

and its inverse different

$$\mathcal{O}^\sharp := \frac{1}{2i\sqrt{2}}\,\mathcal{O}.$$

The attached *Hermitian modular group* is defined by

$$\Gamma_2 := \left\{ M \in \mathcal{O}^{4\times 4}; \quad MJ\overline{M}^{tr} = J \right\}, \quad J = J^{(4)} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad I = I^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the commutator subgroup $\mathcal{C}\Gamma_2$ has the index 2 in Γ_2, there exists exactly one non-trivial abelian character ν on Γ_2, which extends the non-trivial character of $\mathrm{Sp}_2(\mathbb{Z})$ (cf. [D1]). If σ is an arbitrary abelian character and $k \in \mathbb{Z}$, the vector space $[\Gamma_2, k, \sigma]$ of all *Hermitian modular forms* of weight k and character σ consists of all holomorphic functions $F : H_2(\mathbb{C}) \to \mathbb{C}$ satisfying

$$f|_k M(Z) := \det(CZ + D)^{-k} \cdot f(M\langle Z\rangle) = \sigma(M) \cdot f(z)$$

for all $M \in \Gamma_2$ (cf. [Br]). The subspace $[\Gamma_2, k, \sigma]_0$ of *cusp forms* is characterized by the condition

$$f|\Phi \equiv 0,$$

where Φ stands for the Siegel Φ-operator. The superscript *sym* resp. *skew* denotes the subspace of symmetric resp. skew-symmetric modular forms characterized by

$$f \circ I_{tr} = f \quad \text{resp.} \quad f \circ I_{tr} = -f,$$

where $I_{tr}(Z) = Z^{tr}$. Examples are given by the Eisenstein series

$$E_k \in [\Gamma_2, k, 1]^{sym}, \quad E_k|\Phi^2 = 1, \quad k \geq 4 \text{ even}$$

(cf. [DK], in particular for $k = 4$). Each $F \in [\Gamma_2, k, 1]$ possesses a Fourier and a Fourier-Jacobi expansion of the form

$$f(Z) = \sum_{\substack{T=\left(\begin{smallmatrix} n & t \\ \bar{t} & m \end{smallmatrix}\right)\geq 0 \\ n,m\in\mathbb{N}_0, t\in\mathcal{O}^\sharp}} \alpha_F(T) e^{2\pi i \mathrm{trace}(TZ)} = \sum_{m=0}^{\infty} \varphi_m(\tau, z, w) e^{2\pi i m\tau'}.$$

We define the *Maaß space* $\mathcal{M}_k$ to consist of all Maaß lifts of Jacobi forms of weight k and index 1 just as in [S], [H].

Theorem 1. *a) If $k \geq 4$ is even one has*

$$E_k \in \mathcal{M}_k, \quad \mathcal{M}_k \subset [\Gamma_2, k, 1]^{sym} \quad \text{and} \quad \dim \mathcal{M}_k = \frac{k}{2} - 1.$$

b) If $k \geq 5$ is odd one has

$$\mathcal{M}_k \subset [\Gamma_2, k, 1]^{skew} \quad \text{and} \quad \dim \mathcal{M}_k = \left[\frac{k-3}{6}\right].$$

In particular there exist unique $G_k \in \mathcal{M}_k$ with

$$\mathcal{M}_k = \mathbb{C}G_k, \quad \alpha_{G_k}\begin{pmatrix} 1 & i/2\sqrt{2} \\ -i/2\sqrt{2} & 1 \end{pmatrix} = 1, \quad k = 9, 11, 13.$$

Proof. a) Apply [DK], Corollary 2, and the dimension formula in [S] resp. [A], Theorem 5.2. The Maaß lifts are symmetric, because [K2], sect. 4, yields a representation

$$\varphi(\tau, z, w) = \sum_{u:\mathcal{O}^\sharp/\mathcal{O}} f_u(\tau) \cdot \theta_u(\tau, z, w), \quad f_u(\tau) = \sum_{n\in\mathbb{N}_0, n\geq N(u)} \alpha\begin{pmatrix} n & u \\ \overline{u} & 1 \end{pmatrix} e^{2\pi i(n-N(u))\tau},$$

$N(u) := u\overline{u}$, of the first Fourier-Jacobi coefficient as well as the relations

$$\theta_u(\tau, z, w) = \theta_{\overline{u}}(\tau, w, z), \quad f_u(\tau) = f_{\overline{u}}(\tau).$$

b) Apply the dimension formula in [S] resp. [A], Theorem 5.2, and verify that

$$f_{\overline{u}} = -f_u.$$

□

It should be noted that $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = \{0\}$ will be shown later. Considering the first Fourier-Jacobi coefficient φ_1 of $F \in \mathcal{M}_k, k$ odd, we have a representation

$$\varphi_1(\tau, z, w) = f_u \cdot [\theta_u - \theta_{-u}] + f_v \cdot [\theta_v - \theta_{-v}], \quad u := \frac{1}{2i\sqrt{2}}, \quad v := \frac{1}{2} + \frac{1}{2i\sqrt{2}}.$$

Applying [K2], Lemma 4 and Corollary 4, we obtain

$$f_u(\tau+1) = e^{-\pi i/4}\cdot f_u(\tau), \quad f_v(\tau+1) = e^{-3\pi i/4}\cdot f_v(\tau),$$
$$f_u\big|_{k-1}J = \frac{1}{\sqrt{2}}(f_u+f_v), \quad f_v\big|_{k-1}J = \frac{1}{\sqrt{2}}(f_u-f_v), \quad J=J^{(2)}.$$

Define a character $\rho : \Gamma_0(4)\to\mathbb{C}^*$ by

$$\rho\begin{pmatrix}1&1\\0&1\end{pmatrix} = e^{-\pi i/4}, \quad \rho\begin{pmatrix}1&0\\4&1\end{pmatrix} = -1, \quad \rho\begin{pmatrix}-1&0\\0&-1\end{pmatrix} = 1,$$

as well as

$$\mathcal{A}_{k-1} := \left\{g\in[\Gamma_0(4),\ k-1,\ \rho^3]; \quad g-\sqrt{2}g\big|_{k-1}J\in[\Gamma_0(4),\ k-1,\ \rho]\right\}.$$

Let $\eta(\tau)$ denote the Dedekind eta-function and $\theta(\tau)=\sum_{n\in\mathbb{Z}}e^{\pi i n^2\tau}$ the standard theta-function.

Corollary 1. *Let $k\in\mathbb{N}$ be odd. Then the map*

$$\mathcal{M}_k\to\mathcal{A}_{k-1}, \quad F\mapsto f_v,$$

is an isomorphism and one has

$$\begin{aligned}\mathcal{A}_{k-1} &= \eta^{15}(\tau)\theta(2\tau)\cdot[SL_2(\mathbb{Z}),k-9,1]\\ &\oplus\eta^{15}(\tau)\theta(2\tau)\cdot\left[\theta^4(2\tau)-80\frac{\eta^8(4\tau)}{\eta^4(2\tau)}\right]\cdot[SL_2(\mathbb{Z}),k-11,1].\end{aligned}$$

Proof. Only the description of $\mathcal{A}_{k-1}$ remains to be proved. Therefore apply [C]. The functions above belong to $\mathcal{A}_{k-1}$ and the sum is direct because the non-trivial quotients are not invariant under $SL_2(\mathbb{Z})$. Thus the claim follows from the dimension formula in Theorem 1. □

Next we consider the non-trivial character ν on Γ_2. Hence every $F\in[\Gamma_2,k,\nu]$ possesses a Fourier- and Fourier-Jacobi expansion of the form

$$F(Z) = \sum_{\substack{T=\begin{pmatrix}n&t\\\overline{t}&m\end{pmatrix}\geq 0\\ n,m\in\frac{1}{2}+\mathbb{N}_0,\,t\in\frac{1}{4}+\mathcal{O}^\sharp}}\alpha_F(T)e^{2\pi i\,\mathrm{trace}\,(TZ)} = \sum_{\substack{m=1\\ m\ odd}}^{\infty}\varphi_{m/2}(\tau,z,w)e^{\pi i m\tau^2}.$$

Let $\mathcal{M}_k^*$ denote the Maaß space consisting of all lifts of Jacobi forms of index $\frac{1}{2}$ similar to [M] and [K3]. Each Jacobi form of index $\frac{1}{2}$ possesses a representation

$$\begin{aligned}\varphi_{1/2}(\tau,z,w) &= f_u^*(\tau)\cdot\theta_u^*(\tau,z,w)+f_v^*(\tau)\cdot\theta_v^*(\tau,z,w),\\ f_a^*(\tau) &= \sum_{n=0}^{\infty}\alpha_F\begin{pmatrix}n+1/2 & -a/i\sqrt{2}\\ \overline{a}/i\sqrt{2} & 1/2\end{pmatrix}e^{2\pi i(n-N(a)+1/2)\tau},\\ \theta_a^*(\tau,z,w) &= -i\sum_{g\in\frac{a}{-i\sqrt{2}}+\mathcal{O}}e^{\pi i(N(g)\tau+\overline{g}z+gw+Re(g))}.\end{aligned}$$

The behavior of θ_u^*, θ_v^* under $SL_2(\mathbb{Z})$ can be obtained from [H]. Hence we get

$$\begin{aligned} f_u^*\big|_{k-1}J &= \frac{1}{\sqrt{2}}\left(f_u^* + f_v^*\right), \quad f_v^*\big|_{k-1}J = \frac{1}{\sqrt{2}}(f_u^* - f_v^*), \quad J = J^{(2)}, \\ f_v^* \in \mathcal{A}_{k-1}^* &= \left\{g \in \left[\Gamma_0(4), k-1, \rho^{-1}\right]; \quad g + \sqrt{2}g\big|_{k-1}J \in \left[\Gamma_0(4), k-1, \rho^{-3}\right]\right\}. \end{aligned}$$

Theorem 2. *One has $\mathcal{M}_k^* = \{0\}$, if k is even. If $k \in \mathbb{N}$ is odd $\mathcal{M}_k^* \subset [\Gamma_2, k, \nu]^{sym}$ holds. The map*

$$\mathcal{M}_k^* \to \mathcal{A}_{k-1}^*, \quad F \mapsto f_v^*,$$

is an isomorphism and one has

$$dim\, \mathcal{M}_k^* = \left[\frac{k+3}{6}\right].$$

In particular there exist unique modular forms $F_k \in [\Gamma_2, k, \nu]^{sym}$ satisfying

$$\mathcal{M}_k^* = \mathbb{C}F_k, \quad \alpha_{F_k}\begin{pmatrix} 1/2 & (1+i\sqrt{2})/4 \\ (1-i\sqrt{2})/4 & 1/2 \end{pmatrix} = 1, \quad k = 3, 5, 7.$$

Proof. Only the dimension formula remains to be proved. Therefore check that the mapping

$$\mathcal{A}_{k-1}^* \to \mathcal{A}_{k+5}, \quad g(\tau) \mapsto \eta^{12}(\tau) \cdot g(\tau),$$

is an isomorphism and use Theorem 1b). □

Note that the Fourier expansions of F_3, F_5, F_7 and G_9, G_{11}, G_{13} can easily be computed by means of Corollary 1.

3. Borcherds products

It was pointed out in [DK] how to construct Borcherds products, which are Hermitian modular forms with respect to the extended modular group $\tilde{\Gamma}_2 = \langle \Gamma_2 \cup \{I_{tr}\}\rangle$. The obstruction space $[SL_2(\mathbb{Z}), 3, \overline{\rho}_L]$, where ρ_L is the Weil representation, has the dimension 2. It is spanned by the Eisenstein series

$$e_3(\tau) = 1 - 2\cdot\frac{2}{3}e^{\pi i\tau/4} - \frac{10}{3}e^{\pi i\tau/2} - 2\cdot\frac{20}{3}e^{3\pi i\tau/4} - \frac{34}{3}e^{\pi i\tau} - \frac{100}{3}e^{3\pi i\tau/2} - \frac{130}{3}e^{2\pi i\tau} + \ldots$$

(cf. [DK], section 3) and the theta series

$$\begin{aligned} \Theta(\tau) &= \sum_{\lambda\in\mathcal{O}^\sharp}(\lambda^2 + \overline{\lambda}^2)e^{2\pi i N(\lambda)\tau} \\ &= -2\cdot\frac{1}{4}e^{\pi i\tau/4} + e^{\pi i\tau/2} + 2\cdot\frac{1}{2}e^{3\pi i\tau/4} - 2e^{\pi i\tau} - 2e^{3\pi i\tau/2} + 4e^{2\pi i\tau} + \ldots \end{aligned}$$

where the components can be recovered from these functions. Hence it suffices to verify by a simple calculation that the main parts below satisfy Borcherds obstruction condition

[B2], Theorem 3.1:

$$\begin{array}{rrr} & e^{-\pi i\tau/2}+2\cdot 2e^{-\pi i\tau/4} & +6 \\ & 2\cdot e^{-3\pi i\tau/4}+2\cdot 2e^{-\pi i\tau/4} & +16 \\ & e^{-\pi i\tau}+2e^{-\pi i\tau/2} & +18 \\ & e^{-3\pi i\tau/2}+2e^{-\pi i\tau/2} & +40 \\ e^{-2\pi i\tau}- & e^{-\pi i\tau/2}+2\cdot 6e^{-\pi i\tau/4} & +48. \end{array}$$

We obtain Borcherds products ϕ_k with respect to $\tilde{\Gamma}_2$, which have zeros along rational quadratic divisors with discriminants ≤ 8. Since $\tilde{\Gamma}_2$ acts transitively on the set of rational quadratic divisors of fixed discriminant (cf. [FH]) it suffices to consider the following representatives:

$$\begin{aligned}
\mathcal{H}_1 &= \{Z \in H_2(\mathbb{C});\ z = w\} && = H_2(\mathbb{R}), \\
\mathcal{H}_2 &= \{Z \in H_2(\mathbb{C});\ w = -z\} && = \{Z \left[\begin{smallmatrix} 1 & 0, \\ 0 & i\sqrt{2} \end{smallmatrix}\right];\ Z \in H_2(\mathbb{R})\}, \\
\mathcal{H}_3 &= \left\{Z \in H_2(\mathbb{C});\ w = \left(-\tfrac{1}{3} + i\tfrac{2}{3}\sqrt{2}\right) z\right\} && = \left\{Z \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1-i\sqrt{2} \end{smallmatrix}\right]; Z \in H_2(\mathbb{R})\right\}, \\
\mathcal{H}_4 &= \left\{Z \in H_2(\mathbb{C});\ w = z - i\sqrt{2}\right\} && = \left\{Z + \tfrac{1}{2}\left(\begin{smallmatrix} 0 & i\sqrt{2} \\ -i\sqrt{2} & 0 \end{smallmatrix}\right);\ Z \in H_2(\mathbb{R})\right\}, \\
\mathcal{H}_6 &= \left\{Z \in H_2(\mathbb{C});\ w = \left(\tfrac{1}{3} - i\tfrac{2}{3}\sqrt{2}\right) z\right\} && = \left\{Z \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2+i\sqrt{2} \end{smallmatrix}\right];\ Z \in H_2(\mathbb{R})\right\}, \\
\mathcal{H}_8 &= \{Z \in H_2(\mathbb{C});\ w = 1 - z\} && = \left\{Z \left[\begin{smallmatrix} 1 & 0 \\ 0 & i\sqrt{2} \end{smallmatrix}\right] + \tfrac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right); Z \in H_2(\mathbb{R})\right\},
\end{aligned}$$

where $A[B] := \overline{B}^{tr} AB$. Thus [DK], Theorem 5, yields

Theorem 3. *There exist Hermitian modular forms $\phi_k \in [\mathcal{C}\Gamma_2, k, 1], k = 3, 8, 9, 20, 24$, whose orders of zeros along the rational quadratic divisors are given by the following table.*

	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_6$	$\mathcal{H}_8$
$\phi_3 \in [\Gamma_2, 3, \nu]^{sym}$	2	1	0	0	0	0
$\phi_8 \in [\Gamma_2, 8, 1]^{sym}$	2	0	1	0	0	0
$\phi_9 \in [\Gamma_2, 9, 1]^{skew}$	1	2	0	1	0	0
$\phi_{20} \in [\Gamma_2, 20, 1]^{sym}$	0	2	0	0	1	0
$\phi_{24} \in [\Gamma_2, 24, \nu]^{sym}$	6	0	0	0	0	1

Note that we have the following descriptions in terms of transformations from $\tilde{\Gamma}_2$:

$$\begin{aligned}
\mathcal{H}_1 &= \left\{Z \in H_2(\mathbb{C});\quad Z = Z^{tr}\right\}, \\
\mathcal{H}_2 &= \left\{Z \in H_2(\mathbb{C});\quad Z = \left(Z^{tr}\right)\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right\}, \\
\mathcal{H}_4 &= \left\{Z \in H_2(\mathbb{C});\quad Z = Z^{tr} + \left[\begin{smallmatrix} 0 & i\sqrt{2} \\ -i\sqrt{2} & 0 \end{smallmatrix}\right]\right\}, \\
\mathcal{H}_8 &= \left\{Z \in H_2(\mathbb{C});\quad Z = \left(Z^{tr}\right)\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] + \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\}.
\end{aligned}$$

By virtue of these modular transformations Hermitian modular forms have got necessary zeros described in the following

Lemma 1. *a) If $F \in [\mathcal{C}\Gamma_2, k, 1]^{skew}$ then $F = 0$ on $\mathcal{H}_1$ and $F = 0$ on $\mathcal{H}_4$.*
b) If $F \in [\mathcal{C}\Gamma_2, k, 1]^{sym}, k$ odd, or $F \in [\mathcal{C}\Gamma_2, k, 1]^{skew}, k$ even, then $F = 0$ on $\mathcal{H}_2$.
c) If $F \in [\Gamma_2, k, 1]^{sym}, k$ odd, or $F \in [\Gamma_2, k, 1]^{skew}, k$ even, then $F = 0$ on $\mathcal{H}_8$.
d) If $F \in [\Gamma_2, k, \nu]^{sym}, k$ even, or $F \in [\Gamma_2, k, \nu]^{skew}, k$ odd, then $F = 0$ on $\mathcal{H}_8$.

Proof. Apply the description of $\mathcal{H}_j$ above and use the fact that

$$\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \in \mathcal{C}\Gamma_2, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} I & H \\ 0 & I \end{pmatrix} \in \mathcal{C}\Gamma_2, \quad H = \begin{pmatrix} 0 & i\sqrt{2} \\ -i\sqrt{2} & 0 \end{pmatrix},$$
$$\begin{pmatrix} I & H \\ 0 & I \end{pmatrix} \notin \mathcal{C}\Gamma_2, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

due to [D1]. □

A simple application concerns zeros of higher order.

Corollary 2. *Let $F \in [\mathcal{C}\Gamma_2, k, 1]^{sym}$, $s \in \mathbb{N}_0$.*
a) If $ord\, F \geq 2s + 1$ on $\mathcal{H}_1 = H_2(\mathbb{R})$, then $ord\, F \geq 2s + 2$ on $H_2(\mathbb{R})$.
b) If k is even and $ord\, F \geq 2s + 1$ on $\mathcal{H}_2$, then $ord\, F \geq 2s + 2$ on $\mathcal{H}_2$.

Proof. a) Theorem 3 yields

$$G := F\phi_9\phi_{20}^s/\phi_3^{s+1} \in [\mathcal{C}\Gamma_2, k + 6 + 17s, 1]^{skew}.$$

One has $G = 0$ on $H_2(\mathbb{R})$ due to Lemma 1 and therefore it follows that $ord\, F \geq 2s + 2$ on $H_2(\mathbb{R})$.
b) Consider $H := F(\phi_8/\phi_3)^{2s+1} \in [\mathcal{C}\Gamma_2, k + 5(2s + 1), 1]^{sym}$. □

4. The structure theorem

In our case it does not seem to be possible to apply a reduction process along $\mathcal{H}_1 = H_2(\mathbb{R})$ as in [DK]. In our approach we replace $\mathcal{H}_1$ by $\mathcal{H}_3$ as well as $\mathcal{H}_2$ and apply the structure results from [D3] resp. [IO].
Let $t = 2$, $a = i\sqrt{2}$ or $t = 3$, $a = 1 - i\sqrt{2}$ and $M_t := \operatorname{diag}(1, \overline{a}, 1, a^{-1})$, hence $\mathcal{H}_t = M_t\langle H_2(\mathbb{R})\rangle$. This yields an embedding of the paramodular group $Sp\left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ of level t and of its maximal discrete extension Γ_t^* of index 2 into the extended Hermitian modular group $\widetilde{\Gamma}_2$ (cf. [Kö], [DM]). Given $F \in [\Gamma_2, k, \sigma]$ with $F(Z^{tr}) = \varepsilon \cdot F(Z)$ we consider the attached Witt vector

$$W_tF : H_2(\mathbb{R}) \to \mathbb{C}, \quad Z \mapsto F(M_t\langle Z\rangle), \qquad \text{in} \quad [\Gamma_t^*, k, \sigma^*],$$

where $\sigma^* : \Gamma_t^* \to \{\pm 1\}$ is the character uniquely determined by

$$\sigma^* \begin{pmatrix} I & H \\ 0 & I \end{pmatrix} = \sigma \begin{pmatrix} I & H \\ 0 & I \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma^*(V_t^*) = \varepsilon \cdot \sigma \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Starting with $F \in [\Gamma_2, k, 1]^{sym}$ we obtain $W_t F \in [\Gamma_t^*, k, 1]$.

Note that

$$(1) \qquad F_5|_{\mathcal{H}_3} \not\equiv 0,$$

since $F_5 = 0$ on $\mathcal{H}_3$ as well as $F_5 = 0$ on $\mathcal{H}_2$ due to Lemma 1 would imply

$$F_5 \phi_3 / \phi_8 \in [\Gamma_2, 0, 1]^{sym} = \mathbb{C}$$

with a zero along $\mathcal{H}_2$. Thus this constant would be 0. On the other hand $F_5 \equiv 0$ contradicts Theorem 2.

Lemma 2. *The graded ring* $\bigoplus_{k \in 2\mathbb{Z}} [\Gamma_3^*, k, 1]$ *is generated by*

$$W_3(E_4),\ W_3(E_6),\ W_3(\phi_3^2),\ W_3(\phi_3 F_5),\ W_3(F_5^2),\ W_3(E_{12}).$$

Proof. Using [D3], section 5, and the notations there except $E_k^{(3)}$ for the Eisenstein series in $[\Gamma_3^*, k, 1]$ we obtain

$$\begin{array}{lll}
(2) & W_3(\phi_3) \in [\Gamma_3^*, 3, \kappa\chi]_0 & = \mathbb{C}\psi_1^3, \\
& W_3(E_4) \in [\Gamma_3^*, 4, 1] & = \mathbb{C}E_4^{(3)}, \\
(3) & W_3(F_5) \in [\Gamma_3^*, 5, \kappa\chi]_0 & = \mathbb{C}\psi_1 f_4, \\
& W_3(E_6) \in [\Gamma_3^*, 6, 1] & = \mathbb{C}E_6^{(3)} + \mathbb{C}\psi_1^6, \\
& W_3(E_{12}) \in [\Gamma_3^*, 12, 1] & = \mathbb{C}E_{12}^{(3)} + \mathbb{C}{E_6^{(3)}}^2 + \mathbb{C}{E_4^{(3)}}^3 + \mathbb{C}E_6^{(3)}\psi_1^6 + \mathbb{C}E_4^{(3)}\psi_1^4 f_4.
\end{array}$$

Hence $E_4^{(3)}$, $E_6^{(3)}$, ψ_1^6, $\psi_1^4 f_4$, $\psi_1^2 f_4^2$, $E_{12}^{(3)}$ appear as polynomials in the modular forms quoted above, where one has to use a calculation on the Fourier coefficients in [K2] in order to obtain $E_{12}^{(3)}$. Then the result follows from [D3], Lemma 5.3. $\square$

An easy consequence is

Corollary 3. *a) Let* $F \in [\Gamma_2, k, 1]^{sym}, k$ *even. Then there exists an isobaric polynomial* P *such that*

$$\left(F - P(E_4, E_6, \phi_3^2, \phi_3 F_5, F_5^2, E_{12})\right) \frac{\phi_3}{\phi_8} \in [\Gamma_2, k-5, \nu]^{sym}.$$

b) Let $F \in [\Gamma_2, k, \nu]^{sym}, k$ *odd. Then there exist isobaric polynomials* P_1 *and* P_2 *such that*

$$\left(F - \phi_3 \cdot P_1(E_4, E_6, \phi_3^2, \phi_3 F_5, F_5^2, E_{12}) - F_5 \cdot P_2(E_4, E_6, E_{12})\right) \frac{\phi_3}{\phi_8} \in [\Gamma_2, k-5, 1]^{sym}$$

with zeros of order ≥ 2 *along* $\mathcal{H}_2$.

Proof. a) According to Lemma 2 we can choose P such that

$$F - P(E_4, E_6, \phi_3^2, \phi_3 F_5, F_5^2, E_{12}) = 0 \quad \text{on } \mathcal{H}_3.$$

Then Theorem 3 completes the proof.
b) Proceed as before. It follows from Lemma 2, (2) and (3) as well as the cases 1 and 4 in the proof of [D3], Theorem 5.3 a), that P_1 and P_2 exist satisfying

$$F - \phi_3 \cdot P_1(E_4, E_6, \phi_3^2, \phi_3 F_5, F_5^2, E_{12}) - F_5 \cdot P_2(E_4, E_6, E_{12}) = 0 \quad \text{on } \mathcal{H}_3.$$

Then Theorem 3 and Corollary 2 complete the proof. □

This allows us to describe the spaces in the cases of small weights. Using the reduction process in Corollary 3 as well as Theorem 1 and 2 we get

Lemma 3. *a) If* $k = 2, 4, 6, 8, 10$ *one has* $[\Gamma_2, k, 1]^{sym} = \mathcal{M}_k$*, in particular*

$$\begin{aligned} [\Gamma_2, 2, 1]^{sym} &= \{0\}, \quad [\Gamma_2, 4, 1]^{sym} = \mathbb{C}E_4, \quad [\Gamma_2, 6, 1]^{sym} = \mathbb{C}E_6 + \mathbb{C}\phi_3^2, \\ [\Gamma_2, 8, 1]^{sym} &= \mathbb{C}E_4^2 + \mathbb{C}\phi_3 F_5 + \mathbb{C}\phi_8, \\ [\Gamma_2, 10, 1]^{sym} &= \mathbb{C}E_4 E_6 + \mathbb{C}E_4 \phi_3^2 + \mathbb{C}F_5^2 + \mathbb{C}F_{10}, \quad F_{10} := \phi_8 F_5/\phi_3. \end{aligned}$$

b) If $k = 1, 3, 5, 7$ *one has* $[\Gamma_2, k, \nu]^{sym} = \mathcal{M}_k^*$*, in particular*

$$\begin{aligned} [\Gamma_2, 1, \nu]^{sym} &= \{0\}, \quad [\Gamma_2, 3, \nu]^{sym} = \mathbb{C}\phi_3 = \mathbb{C}F_3, \quad [\Gamma_2, 5, \nu]^{sym} = \mathbb{C}F_5, \\ [\Gamma_2, 7, \nu]^{sym} &= \mathbb{C}\phi_3 E_4 = \mathbb{C}F_7. \end{aligned}$$

At this point we also have to consider the restrictions on $\mathcal{H}_2$.

Lemma 4. $E_4 E_6|_{\mathcal{H}_2}$ *and* $F_{10}|_{\mathcal{H}_2}$ *are linearly independent.*

Proof. Lemma 3 yields $\{F|_{\mathcal{H}_2};\ F \in [\Gamma_2, 10, 1]^{sym}\} = \mathbb{C}E_4 E_6|_{\mathcal{H}_2} + \mathbb{C}F_{10}|_{\mathcal{H}_2}$. Now proceed in analogy with [DK], Theorem 1. According to [GN] the Maaß space in $[\Gamma_2^*, 10, 1]$ consists of the lifts of the classical space of Jacobi forms of weight 10 and index 2, which has the dimension 2 due to [EZ]. □

An application yields

Corollary 4. *The graded ring* $\bigoplus_{k \in 2\mathbb{Z}} [\Gamma_2^*, k, 1]$ *is generated by*

$$W_2(E_4),\ W_2(E_6),\ W_2(\phi_8),\ W_2(F_{10}) \quad \textit{and} \quad W_2(E_{12}).$$

Proof. The restrictions do not vanish, which follows from Lemma 4 for F_{10}. Using the notation of [IO] with a superscript (2) we obtain

$$\begin{aligned} W_2(E_k) &\in [\Gamma_2^*, k, 1] = \mathbb{C}F_k^{(2)}, \quad k = 4, 6, \\ W_2(\phi_8) &\in [\Gamma_2^*, 8, 1]_0 = \mathbb{C}F_8^{(2)}, \\ W_2(F_{10}) &\in [\Gamma_2^*, 10, 1]_0 = \mathbb{C}G_{10}^{(2)}, \\ W_2(E_{12}) &\in [\Gamma_2^*, 12, 1] = \mathbb{C}F_{12}^{(2)} + \mathbb{C}{F_4^{(2)}}^3 + \mathbb{C}{F_6^{(2)}}^3 + \mathbb{C}F_4^{(2)} F_8^{(2)}. \end{aligned}$$

Now one calculates a Fourier coefficient in order to see that $F_{12}^{(2)}$ actually appears in the description of $W_2(E_{12})$ on the right hand side. Thus [IO], Theorem 1, completes the proof. □

Now we have to determine certain vanishing ideals. Let

$$\mathcal{I}_{j,r}(k) := \{F \in [\Gamma_2, k, 1]^{sym};\ F = 0 \quad \text{on } \mathcal{H}_j \text{ of order } \geq r\}$$

and

$$\mathcal{I}_{j,r} := \bigoplus_{k \in 2\mathbb{Z}} \mathcal{I}_{j,r}(k).$$

Since the Eisenstein series are non-cusp forms and $F_5 = 0$ on $\mathcal{H}_2$ due to Lemma 1, Theorem 3, Lemma 3 and 4 yield

$$\begin{aligned} (4) \qquad & \mathcal{I}_{2,2}(2) = \mathcal{I}_{2,2}(4) = \{0\}, \quad \mathcal{I}_{2,2}(6) = \mathbb{C}\,\phi_3^2, \quad \mathcal{I}_{2,2}(8) = \mathbb{C}\,\phi_3 F_5 + \mathbb{C}\,\phi_8, \\ & \mathcal{I}_{2,2}(10) = \mathbb{C}E_4 \cdot \phi_3^2 + \mathbb{C}F_5^2. \end{aligned}$$

This is used in the following

Lemma 5. *There exists a modular form*

$$K_{12} = c_1 E_4 \cdot \phi_3 F_5 + c_2 \phi_3^4 + c_3 E_6 \cdot \phi_3^2 + c_4 E_4^3 + c_5 E_6^2 + c_6 E_{12} + c_7 E_4 \cdot \phi_8 \in [\Gamma_2, 12, 1]^{sym}$$

and $0 \neq \gamma \in \mathbb{C}$ *such that*

$$F_5^3 = \phi_3 \cdot K_{12} + \gamma F_5 \cdot F_{10}, \qquad F_5^2 \cdot F_{10} = \phi_8 \cdot K_{12} + \gamma F_{10}^2.$$

Proof. According to [DK], Theorem 5.2, there exists a relation

$$f_4^3 = \widetilde{c}_1 \psi_1^4 f_4 E_4^{(3)} + \widetilde{c}_2 \psi_1^{12} + \widetilde{c}_3 \psi_1^6 E_6^{(3)} + \widetilde{c}_4 {E_4^{(3)}}^3 + \widetilde{c}_5 {E_6^{(3)}}^2 + \widetilde{c}_6 E_{12}^{(3)}.$$

Due to Lemma 2, (2) and (3) we may choose K_{12} as above such that

$$W_3(F_5^3) = \alpha \psi_1^3 f_4^3 = W_3(\phi_3 \cdot K_{12}).$$

Hence $F_5^3 - \phi_3 \cdot K_{12}$ vanishes on $\mathcal{H}_3$. Then Theorem 3, Lemma 1 and (4) yield

$$(F_5^3 - \phi_3 \cdot K_{12}) \frac{\phi_3}{\phi_8} \in \mathcal{I}_{2,2}(10) = \mathbb{C}E_4 \cdot \phi_3^2 + \mathbb{C}F_5^2.$$

Choosing c_7 properly we obtain $\gamma \in \mathbb{C}$ with

$$F_5^3 = \phi_3 \cdot K_{12} + \gamma F_5 \cdot F_{10}.$$

Multiplication by ϕ_8/ϕ_3 yields the second equation.
If $\gamma = 0$ then the right hand side and therefore F_5 vanish on $\mathcal{H}_1$ and $\mathcal{H}_2$. Theorem 3 and Lemma 3 imply

$$F_5^2/\phi_3 \in [\Gamma_2, 7, \nu]^{sym} = \mathbb{C}\,\phi_3 E_4,$$

hence $F_5^2 = \beta \phi_3^2 E_4$. Therefore it follows that *ord* $F_5 \geq 2$ on $\mathcal{H}_1$ and finally the identity $F_5/\phi_3 \in [\Gamma_2, 2, 1]^{sym} = \{0\}$ yields a contradiction. □

As a consequence of the last argument we have

$$(5) \qquad F_5|_{H_2(\mathbb{R})} \not\equiv 0, \qquad F_{10}|_{H_2(\mathbb{R})} \not\equiv 0.$$

We derive our main result.

Theorem 4. *The graded ring* $R := \bigoplus_{k\in 2\mathbb{Z}}[\Gamma_2, k, 1]^{sym}$ *is generated by*

$$E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ \phi_8,\ F_5^2,\ F_{10} \quad \text{and} \quad E_{12}, \tag{6}$$

where $E_4,\ E_6,\ \phi_3^2,\ \phi_8,\ E_{12}$ *are algebraically independent. All the generators are Maaß lifts. The ideal of cusp forms in* R *is generated by*

$$\phi_3^2,\ \phi_3 F_5,\ \phi_8,\ F_5^2,\ F_{10} \quad \text{and} \quad E_{12} - \frac{441}{691} E_4^3 - \frac{250}{691} E_6^2.$$

The ideal $\mathcal{I}_{2,1} = \mathcal{I}_{2,2}$ *is generated by*

$$\phi_3^2,\ \phi_3 F_5 \quad \text{and} \quad F_5^2.$$

Proof. We show by induction on k that any $F \in [\Gamma_2, k, 1]^{sym}$, k even, is an isobaric polynomial in the modular forms (6) and that

$$\mathcal{I}_{2,2}(k) = \phi_3^2 \cdot [\Gamma_2, k-6, 1]^{sym} + \phi_3 F_5 \cdot [\Gamma_2, k-8, 1]^{sym} + F_5^2 \cdot [\Gamma_2, k-10, 1]^{sym}.$$

This is true for $k \le 10$ by Lemma 3 and (4). Starting with $F \in [\Gamma_2, k, 1]^{sym}$, $k > 10$, we apply Corollary 3 and obtain polynomials P, P_1, P_2 such that

$$G = \left(F - P(E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ F_5^2,\ E_{12})\right) \frac{\phi_3}{\phi_8} \in [\Gamma_2, k-5, \nu]^{sym},$$

$$H = \left(G - \phi_3 \cdot P_1(E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ F_5^2,\ E_{12}) - F_5 \cdot P_2(E_4,\ E_6,\ E_{12})\right) \frac{\phi_3}{\phi_8} \in \mathcal{I}_{2,2}(k-10).$$

The induction hypothesis yields

$$H = \phi_3^2 \cdot H_1 + \phi_3 F_5 \cdot H_2 + F_5^2 \cdot H_3,$$

where $H_1,\ H_2,\ H_3$ are polynomials in the modular forms (6). Hence it follows that

$$\begin{aligned} F = P\left(E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ F_5^2,\ E_{12}\right) + \phi_8 \cdot P_1\left(E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ F_5^2,\ E_{12}\right) \\ + F_{10} \cdot P_2(E_4,\ E_6,\ E_{12}) + \phi_8^2 \cdot H_1 + \phi_8 F_{10} \cdot H_2 + F_{10}^2 \cdot H_3. \end{aligned}$$

Thus R is generated by the modular forms (6).

Now let $F \in \mathcal{I}_{2,2}(k)$, which is an isobaric polynomial in the modular forms (6). By Lemma 5 we may suppose that

$$F = P_1\left(E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ \phi_8,\ F_5^2,\ E_{12}\right) + F_{10} \cdot P_2\left(E_4,\ E_6,\ \phi_3^2,\ \phi_3 F_5,\ \phi_8,\ F_5^2,\ E_{12}\right). \tag{7}$$

Restriction to $\mathcal{H}_2$ yields

$$P_1(E_4,\ E_6,\ 0,\ 0,\ \phi_8,\ 0,\ E_{12}) + F_{10} \cdot P_2(E_4,\ E_6,\ 0,\ 0,\ \phi_8,\ 0,\ E_{12}) = 0 \quad \text{on } \mathcal{H}_2.$$

Hence Corollary 4 and [IO], Theorem 1, imply

$$P_1(X_1,\ X_2,\ 0,\ 0,\ X_5,\ 0,\ X_7) = P_2(X_1,\ X_2,\ 0,\ 0,\ X_5,\ 0,\ X_7) = 0.$$

Thus F has a representation

$$F = \phi_3^2 \cdot H_6 + \phi_3 F_5 \cdot H_8 + F_5^2 \cdot H_{10}, \qquad H_j \in [\Gamma_2, k-j, 1]^{sym}.$$

Hence $\mathcal{I}_{2,2}$ is generated by $\phi_3^2, \phi_3 F_5$ and F_5^2.
Let P be an isobaric polynomial with $P\left(E_4,\, E_6,\, \phi_3^2,\, \phi_8,\, E_{12}\right) = 0$. The restriction to $\mathcal{H}_3$ yields $P\left(E_4,\, E_6,\, \phi_3^2,\, 0,\, E_{12}\right) = 0$ and [D3], Theorem 5.2 implies that $P(X_1,\, X_2,\, X_3,\, 0,\, X_4) = 0$. Thus we have

$$P = \phi_8 \cdot \widetilde{P}\left(E_4,\, E_6,\, \phi_3^2,\, \phi_8,\, E_{12}\right)$$

and an induction shows that $E_4, E_6, \phi_3^2, \phi_8, E_{12}$ are algebraically independent.
In (6) the functions $\phi_3^2,\ \phi_3 F_5,\ \phi_8,\ F_5^2$ and F_{10} are cusp forms. Dealing with E_4, E_6, E_{12} apply the Siegel Φ-operator and use the well-known formula for elliptic Eisenstein series in order to obtain the generators of the ideal of cusp forms quoted above.
The Eisenstein series are Maaß lifts due to Theorem 1 and the other generators according to Lemma 3. □

We quote some other generators in the following

Corollary 5. *The graded ring $R = \bigoplus_{k \in 2\mathbb{Z}} [\Gamma_2, k, 1]^{sym}$ is generated by*

$$E_4,\ E_6,\ E_8,\ E_{10},\ E_{12} \quad \text{and} \quad \phi_3^2,\ \phi_3 F_5,\ F_5^2,$$

where E_4, E_6, E_8, E_{10}, E_{12} are algebraically independent.

Proof. Apply Theorem 4 and Lemma 3. A calculation on the Fourier coefficients in [K2], Corollary 8A, shows that the restrictions of $E_4^2 - E_8$ and $E_4 \cdot E_6 - E_{10}$ to $\mathcal{H}_2$ are non-zero, hence

$$(E_4^2 - E_8)|_{\mathcal{H}_2} = \alpha \phi_8|_{\mathcal{H}_2} \quad \text{and} \quad (E_4 \cdot E_6 - E_{10})|_{\mathcal{H}_2} = \beta F_{10}|_{\mathcal{H}_2}$$

with $\alpha \neq 0$, $\beta \neq 0$.
Due to Igusa's result [Ig] and [DK], Corollary 2, the restrictions of E_4, E_6, E_{10}, E_{12} to $H_2(\mathbb{R})$ are algebraically independent, whereas $(E_4^2 - E_8)|_{H_2(\mathbb{R})} \equiv 0$. Hence the 5 Eisenstein series are algebraically independent. □

Next Corollary 3b and Theorem 4 imply

Corollary 6. *The R-module $\bigoplus_{k\, odd} [\Gamma_2, k, \nu]^{sym}$ is spanned by*

$$\phi_3 \quad \text{and} \quad F_5.$$

We have to investigate vanishing ideals.

Lemma 6. *a) The ideal $\mathcal{I}_{1,1} = \mathcal{I}_{1,2}$ is generated by ϕ_3^2, $\phi_3 F_5$, ϕ_8 and $F_5^2 - \gamma F_{10}$.*
b) The ideal $\mathcal{I}_{3,1}$ is generated by ϕ_8 and F_{10}.
c) The ideal $\mathcal{I}_{3,2}$ is generated by ϕ_8^2, $\phi_8 F_{10}$ and F_{10}^2.

Proof. a) Lemma 5 yields $F_5 \cdot \left(F_5^2 - \gamma F_{10}\right) = \phi_3 \cdot K_{12} = 0$ on $\mathcal{H}_1$. Hence the modular forms above belong to $\mathcal{I}_{1,2}$. Given $F \in \mathcal{I}_{1,2}$ we may reduce F modulo the ideal $\mathcal{I}$ generated by ϕ_3^2, $\phi_3 F_5$, ϕ_8 and $F_5^2 - \gamma F_{10}$. By virtue of (7) we obtain

$$F = P\left(E_4,\, E_6,\, F_5^2,\, E_{12}\right).$$

Since $F_5^2|_{\mathcal{H}_1}$ is a non-trivial Siegel cusp form, the restrictions of E_4, E_6, F_5^2, E_{12} on $\mathcal{H}_1$ are algebraically independent. Hence $F = 0$ follows.
b) Clearly ϕ_8 and $F_{10} = F_5\phi_8/\phi_3$ vanish on $\mathcal{H}_3$. Now apply Lemma 2 and 5 as well as [D3], Lemma 5.3.
c) The mapping $\mathcal{I}_{2,2} \to \mathcal{I}_{3,2}$, $F \mapsto F\phi_8^2/\phi_3^2$, is an isomorphism according to Theorem 3. Now Theorem 4 completes the proof. □

According to the definitions resp. Lemma 5 we have the following relations

$$\begin{aligned}
(\phi_3 F_5)^2 &= \phi_3^2 \cdot F_5^2, & \left(F_5^2\right)^2 &= \phi_3 F_5 \cdot K_{12} + \gamma F_5^2 \cdot F_{10},\\
\phi_3^2 \cdot F_{10} &= \phi_3 F_5 \cdot \phi_8, & \phi_3 F_5 \cdot F_5^2 &= \phi_3^2 \cdot K_{12} \quad + \gamma \phi_3 F_5 \cdot F_{10},\\
\phi_3 F_5 \cdot F_{10} &= \phi_8 \cdot F_5^2, & F_5^2 \cdot F_{10} &= \phi_8 \cdot K_{12} \quad + \gamma F_{10}^2.
\end{aligned}$$

Now define the corresponding polynomials $Q_j \in \mathbb{C}[X_1, \ldots, X_8]$ by

$$\begin{aligned}
Q_1 &= X_4^2 - X_3 X_6, & Q_2 &= X_6^2 - X_4 K - \gamma X_6 X_7,\\
Q_3 &= X_3 X_7 - X_4 X_5, & Q_4 &= X_4 X_6 - X_3 K - \gamma X_4 X_7,\\
Q_5 &= X_4 X_7 - X_5 X_6, & Q_6 &= X_6 X_7 - X_5 K - \gamma X_7^2,
\end{aligned}$$

where $K = c_1 X_1 X_4 + c_2 X_3^2 + c_3 X_2 X_3 + c_4 X_1^3 + c_5 X_2^2 + c_6 X_8 + c_7 X_1 X_5$ due to Lemma 5. We have

$$Q_j\left(E_4,\, E_6,\, \phi_3^2,\, \phi_3 F_5,\, \phi_8,\, F_5^2,\, F_{10},\, E_{12}\right) = 0 \quad \text{for } j = 1, \ldots, 6.$$

Theorem 5. *The graded ring $R = \bigoplus_{k \in 2\mathbb{Z}} [\Gamma_2, k, 1,]^{sym}$ is isomorphic to*

$$\mathbb{C}[X_1, \ldots, X_8]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by $Q_1, \ldots, Q_6$.

Proof. Apply Theorem 4 and the relations above. Hence it suffices to show that any $P \in \mathbb{C}[X_1, \ldots, X_8]$ with $P(E_4, \ldots, E_{12}) \equiv 0$ belongs to $\mathcal{I}$. We may reduce P mod $\mathcal{I}$ and may therefore assume that

$$\begin{aligned}
P = {} & S(X_1, X_2, X_3, X_8) + X_4 \cdot T(X_1, X_2, X_3, X_8)\\
& + \sum_{j \geq 0} (X_6 X_7)^j \cdot \left[X_6 \cdot P_j(X_1, X_2, X_3, X_8) + X_7 \cdot P_j^*(X_1, X_2, X_8)\right]\\
& + \sum_{j \geq 1} (X_6 X_7)^j \cdot P_j^{**}(X_1, X_2, X_8) + X_5 \cdot \widehat{P}(X_1, \ldots, X_8).
\end{aligned}$$

The restriction to $\mathcal{H}_3$ yields

$$S\left(E_4,\, E_6,\, \phi_3^2,\, E_{12}\right) + \phi_3 F_5 \cdot T\left(E_4,\, E_6,\, \phi_3^2,\, E_{12}\right) + F_5^2 \cdot P_0\left(E_4,\, E_6,\, \phi_3^2,\, E_{12}\right) = 0 \quad \text{on } \mathcal{H}_3.$$

Now Lemmma 2 and [D3], Lemma 5.3, imply

$$S(X_1,X_2,X_3,X_8)=T(X_1,X_2,X_3,X_8)=P_0(X_1,X_2,X_3,X_8)=0.$$

Next the restriction to $\mathcal{H}_1$ yields

$$F_{10}\cdot P_0^*(E_4,\,E_6,\,E_{12})+\sum_{j\geq 1}\left(F_5^2F_{10}\right)^j\cdot\big[F_5^2\cdot P_j(E_4,\,E_6,\,0,\,E_{12})+F_{10}\cdot P_j^*(E_4,\,E_6,\,E_{12})$$
$$+P_j^{**}(E_4,\,E_6,\,E_{12})\big]=0\quad\text{on }\mathcal{H}_1.$$

According to Lemma 6 the restriction $F_5^2|_{\mathcal{H}_1}=\gamma F_{10}|_{\mathcal{H}_1}$ is a non-trivial Siegel cusp form of weight 10. Hence Igusa's result [Ig] says that

$$P_0^*(X_1,X_2,X_8)=P_j^{**}(X_1,X_2,X_8)=-\gamma P_j(X_1,X_2,0,X_8)+P_j^*(X_1,X_2,X_8)=0.$$

Thus we have got

$$P=\sum_{j\geq 1}(X_6X_7)^j\cdot\left[(X_7-\tfrac{1}{\gamma}X_6)\cdot P_j^*(X_1,X_2,X_8)+X_3\cdot\widetilde{P}_j(X_1,X_2,X_3,X_8)\right]+X_5\cdot\widehat{P}.$$

Now note that

$$X_6X_7(X_7-\tfrac{1}{\gamma}X_6)\equiv-\tfrac{1}{\gamma}X_5K \bmod \mathcal{I},\quad X_3X_6X_7\equiv X_4X_5X_7 \bmod \mathcal{I},$$

hence

$$P\equiv X_5\cdot Q(X_1,\dots,X_8)\bmod \mathcal{I}.$$

Thus an induction completes the proof. □

Using Corollary 6 as well as Theorem 4 and 5 we obtain a related structure. According to Lemma 5 we have

$$\gamma F_{10}^2=F_5^2\cdot F_{10}-\phi_8\cdot K_{12},\quad \gamma F_5\cdot F_{10}=F_5^3-\phi_3\cdot K_{12},\quad \phi_3\cdot F_{10}=F_5\cdot\phi_8.$$

Corollary 7. *The graded ring $\bigoplus_{k\in\mathbb{Z}}[\Gamma_2,k,\nu^k]^{sym}$ is generated by*

$$\phi_3,\,E_4,\,F_5,\,E_6,\,\phi_8,\,F_{10},\,E_{12},$$

where ϕ_3, E_4, E_6, ϕ_8, E_{12} are algebraically independent. It is isomorphic to the ring $\mathbb{C}[X_1,\dots,X_7]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by

$$\gamma X_6^2-X_3^2X_6+X_5Q,\quad \gamma X_3X_6-X_3^3+X_1Q,\quad X_1X_6-X_3X_5,$$

where $Q=c_1X_1X_2X_3+c_2X_1^4+c_3X_1^2X_4+c_4X_2^3+c_5X_4^2+c_6X_7+c_7X_2X_5$.

5. Description of R-modules

In this section we describe $\bigoplus_{k\in\mathbb{Z}}[\mathcal{C}\Gamma_2, k, 1]$ as an R-module. At first we need more vanishing ideals of higher order.

Lemma 7. *a) The ideal $\mathcal{I}_{2,2}\cap\mathcal{I}_{3,2}$ is generated by*

$$\phi_3^2\phi_8^2,\ \phi_3F_5\phi_8^2,\ F_5^2\phi_8^2,\ \phi_3F_5F_{10}^2,\ F_5^2F_{10}^2.$$

b) The ideal $\mathcal{I}_{2,4}$ is generated by

$$\phi_3^4,\ \phi_3^3F_5,\ \phi_3^2F_5^2,\ \phi_3F_5^3,\ F_5^4.$$

c) The ideal $\mathcal{I}_{2,4}\cap\mathcal{I}_{3,1}$ is generated by

$$\phi_3^2\phi_8,\ \phi_3F_5\phi_8,\ F_5^2\phi_8,\ F_5^2F_{10}.$$

Proof. a) Let $\mathcal{I}$ denote the ideal generated by the elements quoted above. Then $\mathcal{I}\subset\mathcal{I}_{2,2}\cap\mathcal{I}_{3,2}$ follows from Lemma 6 and Theorem 3. Now let $F\in\mathcal{I}_{2,2}\cap\mathcal{I}_{3,2}$. We may reduce F by $\mathcal{I}$ in order to show that $F+\mathcal{I}=\mathcal{I}$. According to Lemma 6 we have got a representation

$$F=\phi_8^2\cdot G_1+\phi_8F_{10}\cdot G_2+F_{10}^2\cdot G_3,$$

where $G_j = P_j\left(E_4,\, E_6,\, \phi_3^2,\, \phi_3F_5,\, \phi_8,\, F_5^2,\, F_{10},\, E_{12}\right)$ due to Theorem 3. By virtue of (7) we may suppose that P_j is of order ≤ 1 in F_{10}. Shifting to the other factors allows us to assume that F_{10} does not occur in P_1 and P_2. Lemma 5 yields $F_{10}^3 = \frac{1}{\gamma}\left(F_5^2F_{10}^2-\phi_8F_{10}K_{12}\right)$. Thus we may suppose that F_{10} does not occur in P_3 and

$$\begin{aligned}F=\phi_8^2\cdot Q_1(E_4,\,E_6,\,\phi_8,\,E_{12})+\phi_8F_{10}\cdot Q_2(E_4,\,E_6,\,\phi_8,\,E_{12})\\ +\left(\phi_8K_{12}-F_5^2F_{10}\right)\cdot Q_3(E_4,\,E_6,\,\phi_8,\,E_{12}).\end{aligned}$$

Now Lemma 5 leads to

$$\begin{aligned}F=\phi_8^2\cdot Q_1(E_4,\,E_6,\,\phi_8,\,E_{12})+\phi_8F_{10}\cdot Q_2(E_4,\,E_6,\,\phi_8,\,E_{12})\\ +\phi_8\left(c_4E_4^3+c_5E_6^2+c_6E_{12}+c_7\phi_8E_4\right)\cdot Q_3(E_4,\,E_6,\,\phi_8,\,E_{12})=0\quad\text{on }\mathcal{H}_2.\end{aligned}$$

Then Corollary 4 and [IO], Theorem 1, imply that this relation can only trivially be fulfilled, i.e.

$$Q_2=0,\quad X_3^2Q_1+X_3\left(c_4X_1^3+c_5X_2^2+c_6X_4+c_7X_3X_4\right)Q_3=0.$$

Hence we have

$$F=\phi_8\cdot\left(c_1\phi_3F_5E_4+c_2\phi_3^4+c_3\phi_3^2E_6-F_5^2F_{10}\right)\cdot Q_3(E_4,\,E_6,\,\phi_8,\,E_{12}).$$

Another reduction mod $\mathcal{I}$ allows us to assume $Q_3=Q(E_4,\,E_6,\,E_{12})$ as well as

$$\left(c_1\phi_3F_5E_4+c_2\phi_3^4+c_3\phi_3^2E_6\right)\cdot Q(E_4,\,E_6,\,E_{12})=0\quad\text{on }\mathcal{H}_3.$$

Now Lemma 2 and [D3], Theorem 5.3, show that this equation must be trivial, i.e. $Q(E_4,\, E_6,\, E_{12}) = 0$ and $F = 0$.
b) The mapping $\mathcal{I}_{2,2} \cap \mathcal{I}_{3,2} \to \mathcal{I}_{2,4}$, $F \mapsto F\phi_3^2/\phi_8^2$, is an isomorphism.
c) Proceed just as in a). $\square$

As a first application we determine $\bigoplus_{k\in\mathbb{Z}}[\mathcal{C}\Gamma_2, k, 1]^{sym}$ completely.

Corollary 8. *a) The R-module $\bigoplus_{k\,odd}[\Gamma_2, k, 1]^{sym}$ is spanned by*

$$\phi_{24}\phi_3 \quad and \quad \phi_{24}F_5\,(F_5/\phi_3)^j\,, \quad j = 0, 1, 2, 3.$$

b) The R-module $\bigoplus_{k\,even}[\Gamma_2, k, \nu]^{sym}$ is spanned by

$$\phi_{24}\,(F_5/\phi_3)^j\,, \quad j = 0, 1, 2, 3.$$

Proof. a) Let $F \in [\Gamma_2, k, 1]^{sym}$, k odd. Lemma 1 yields $F = 0$ on $\mathcal{H}_2$ and $\mathcal{H}_8$. We obtain

$$F\phi_3^3/\phi_{24} \in \mathcal{I}_{2,4}(k-15)$$

from Theorem 3. Now Lemma 7 completes the proof.
b) Let $F \in [\Gamma_2, k, \nu]^{sym}$, k even. Lemma 1 yields $F = 0$ on $\mathcal{H}_8$. Consider

$$F\phi_3^2\phi_8/\phi_{24} \in \mathcal{I}_{2,2}(k-10) \cap \mathcal{I}_{3,1}(k-10)$$

due to Theorem 3. Now Lemma 7 completes the proof again. $\square$

We add a few

Remarks. a) The restrictions of F_5 and $\phi_{24}F_5^3/\phi_3^2$ to $H_2(\mathbb{R})$ do not vanish identically. Hence $\left\{F|_{H_2(\mathbb{R})};\ F \in [\mathcal{C}\Gamma_2, k, 1]^{sym}\right\}$ coincides with the space of Siegel modular forms of weight k with respect to the commutator subgroup $\mathcal{C}Sp_2(\mathbb{Z})$ (cf. [Ig]).
b) The restrictions $F|_{\mathcal{H}_4}$, $F \in [\mathcal{C}\Gamma_2, k, 1]^{sym}$, yield Siegel modular forms with respect to $\Gamma_0^*(2) < Sp_2(\mathbb{R})$ in the notation of [Ib]. Our results and the appendix in [Ib] show that these restrictions are not surjective in general.

Next we investigate skew-symmetric modular forms.

Lemma 8. *Let $k \in \mathbb{N}$, $k \le 11$ be odd. Then*

$$[\Gamma_2, k, 1]^{skew} = \mathcal{M}_k^*$$

holds, in particular

$$[\Gamma_2, k, 1]^{skew} = \{0\}, \quad k = 1, 3, 5, 7,$$
$$[\Gamma_2, 9, 1]^{skew} = \mathbb{C}\phi_9 = \mathbb{C}G_9, \qquad [\Gamma_2, 11, 1]^{skew} = \mathbb{C}G_{11}.$$

Proof. Given $F \in [\Gamma_2, k, 1]^{skew}$ one has $F = 0$ on $\mathcal{H}_1$ and $\mathcal{H}_4$ due to Lemma 1. Then $F\phi_3^2/\phi_9 \in \mathcal{I}_{1,4}(k-3)$ follows. Now apply Lemma 6, $\dim \mathcal{I}_{1,4}(8) \le 1$ as well as Theorem 2. $\square$

This allows us to derive

Corollary 9. *a) The R-module* $\bigoplus_{k\,even}[\Gamma_2,k,\nu]^{skew}$ *is spanned by*

$$\phi_3\phi_9, \quad F_5\phi_9, \quad \frac{\phi_8\phi_9}{\phi_3}, \quad (F_5^2-\gamma F_{10})\frac{\phi_9}{\phi_3}.$$

b) The R-module $\bigoplus_{k\,odd}[\Gamma_2,k,\nu]^{skew}$ *is spanned by*

$$\frac{\phi_9\phi_{24}}{\phi_3^2}, \qquad \frac{F_5\phi_9\phi_{24}}{\phi_3^3}.$$

Proof. a) $F\in[\Gamma_2,k,\nu]^{skew}$, k even, vanishes on $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_4$. Hence

$$\frac{\phi_3 F}{\phi_9}\in\mathcal{I}_{1,2}(k-6)$$

follows. Now use Lemma 6.
b) $F\in[\Gamma_2,k,\nu]^{skew}$, k odd, vanishes on $\mathcal{H}_1$, $\mathcal{H}_4$, $\mathcal{H}_8$, hence

$$\frac{\phi_3^3 F}{\phi_9\phi_{24}}\in[\Gamma_2,k-24,\nu]^{sym}.$$

Thus Corollary 8 completes the proof.

□

Finally we obtain

Corollary 10. *a) The R-module* $\bigoplus_{k\,even}[\Gamma_2,k,1]^{skew}$ *is spanned by*

$$\frac{\phi_9\phi_{24}}{\phi_3}, \quad \frac{F_5\phi_9\phi_{24}}{\phi_3^2}, \quad \frac{F_5^2\phi_9\phi_{24}}{\phi_3^3}.$$

b) The R-module $\bigoplus_{k\,odd}[\Gamma_2,k,1]^{skew}$ *is spanned by*

$$\phi_9, \quad G_{11}, \quad \frac{\phi_8^2\phi_9}{\phi_3^2}, \quad (F_5^2-\gamma F_{10})\frac{\phi_8\phi_9}{\phi_3}.$$

Proof. a) $F\in[\Gamma_2,k,1]^{skew}$, k even, vanishes on $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_4$ and $\mathcal{H}_8$ due to Lemma 1. Thus Theorem 3 yields

$$\frac{\phi_3^3 F}{\phi_9\phi_{24}}\in\mathcal{I}_{2,2}(k-24).$$

Now apply Theorem 4.
b) Given $F\in[\Gamma_2,k,1]^{skew}$, k odd, one has $W_3(F)\in[\Gamma_3^*,k,\chi]$ and [D3], Theorem 5.2 yields

$$[\Gamma_3^*,k,\chi]=\psi_1^3\psi_6\cdot[\Gamma_3^*,k-9,1]+\psi_1\psi_4\psi_6\cdot[\Gamma_3^*,k-11,1].$$

In particular these spaces are one-dimensional and spanned by $W_3(\phi_9)$ resp. $W_3(G_{11})$ if $k=9,11$, because $G_{11}=0$ on $\mathcal{H}_3$ would imply

$$\frac{\phi_3 G_{11}}{\phi_8}\in[\Gamma_2,6,\nu]^{skew}=\{0\}$$

due to a) as a contradiction. Using Lemma 2 we find $G \in [\Gamma_2, k-9, 1]^{sym}$ as well as $H \in [\Gamma_2, k-11, 1]^{sym}$ such that

$$F - \phi_9 \cdot G - G_{11} \cdot H = 0 \qquad \text{on} \quad \mathcal{H}_3.$$

Hence Theorem 3 leads to

$$(F - \phi_9 \cdot G - G_{11} \cdot H)\frac{\phi_3}{\phi_8} \in [\Gamma_2, k-5, \nu]^{skew}.$$

Now apply Corollary 9. □

According to the orthogonal relations for characters we have determined $\bigoplus_{k\in\mathbb{Z}}[\mathcal{C}\Gamma_2, k, 1]$ completely (cf. [DK]). Gathering all the terms we obtain a set of generators of $\bigoplus_{k\in\mathbb{Z}}[\mathcal{C}\Gamma_2, k, 1]$, which consists of Hermitian modular forms of weight ≤ 30.

A result of Klingen [Kl] says that the field of symmetric Hermitian modular functions consists of the quotients of two symmetric Hermitian modular forms of the same even weight. Hence Theorem 4 and 3 as well as Lemma 5 imply

Corollary 11. *a) The field $\mathcal{H}_K^{sym}$ of symmetric Hermitian modular functions with respect to Γ_2 is generated by*

$$\psi_1 = \frac{\phi_3^2}{E_6}, \quad \psi_2 = \frac{\phi_8}{E_4^2}, \quad \psi_3 = \frac{E_6^2}{E_4^3}, \quad \psi_4 = \frac{E_{12}}{E_4^3} \qquad \textit{and} \quad \psi_5 = \frac{\phi_3 F_5}{E_4^2},$$

where ψ_1, ψ_2, ψ_3, ψ_4 are algebraically independent and where ψ_5 satisfies an equation of degree 3 over $\mathbb{C}(\psi_1, \ldots, \psi_4)$, namely

$$\psi_5^3 - \gamma\psi_2 \cdot \psi_5^2 - c_1\psi_1^2\psi_3 \cdot \psi_5 + \psi = 0$$

with

$$\psi = c_2\psi_1^4\psi_3^2 + c_3\psi_1^3\psi_3^2 + c_4\psi_1^2\psi_3 + c_5\psi_1^2\psi_3^2 + c_6\psi_1^2\psi_3\psi_4 + c_7\psi_1^2\psi_2\psi_3.$$

b) The field of Hermitian modular functions with respect to $\mathcal{C}\Gamma_2$ is an extension of degree 4 over $\mathcal{H}_K^{sym}$ generated by

$$\psi_6 = \frac{\phi_3^8}{\phi_{24}}, \qquad \psi_6 \circ M = \nu(M) \cdot \psi_6, \quad M \in \Gamma_2, \qquad \psi_6 \circ I_{tr} = \psi_6,$$
$$\psi_7 = \frac{\phi_9\phi_3^5}{\phi_{24}}, \qquad \psi_7 \circ M = \psi_7, \quad M \in \Gamma_2, \qquad \psi_7 \circ I_{tr} = -\psi_7.$$

6. Theta series

We obtain theta series

$$\vartheta_p(Z) := \sum_{g \in p + \mathcal{O}^2} e^{2\pi i \bar{g}^{tr} Z g}, \; Z \in H_2(\mathbb{C}), \; p \in \mathcal{O}^{\sharp^2}, \quad Z \in H_2(\mathbb{C}). \tag{8}$$

One has $\vartheta_p = \vartheta_{-p} = \vartheta_{p+h}, h \in \mathcal{O}^2$, and $p + \overline{p} \in \mathcal{O}^2$ yields

$$\vartheta_p(Z^{tr}) = \vartheta_p(Z).$$

Thus the theta series (8) are symmetric Hermitian modular forms of weight 1 and trivial character with respect to the principal congruence subgroup

$$\Gamma_2[\sqrt{-8}] := \{M \in \Gamma_2; \quad M \equiv \pm I \mod \sqrt{-8}\}.$$

The space Θ spanned by these theta series has the dimension 40 as easily checked by comparing the Fourier expansions. The theta transformation formula therefore yields a 40-dimensional representation of the group

$$\Gamma_2/\Gamma_2[\sqrt{8}] \cong \mathrm{PSp}_2(\mathcal{O}/(\sqrt{-8}))$$

of the order $377,487,360 = 2^{23} \cdot 3^2 \cdot 5$. The Molien series of this representation can be calculated to be

$$\frac{1 + \sum_{j=4}^{194} d_j t^{2j}}{(1-t^4)^3 \cdot (1-t^6)^8 \cdot (1-t^8)^{12} \cdot (1-t^{10})^4 \cdot (1-t^{12})^4 \cdot (1-t^{16})^2 \cdot (1-t^{20})^4 \cdot (1-t^{24})^3},$$

$d_j \in \mathbb{N}_0$. Thus the non-zero Hermitian modular forms in $[\Gamma_2, k, 1]^{sym}, k$ odd, cannot be represented as polynomials in the functions from Θ.

Consider the half-space $H_2(\mathbb{H})$ of quaternions of degree 2 (cf. [K1]) and the Hurwitz order

$$\Lambda = \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}\omega, \quad \omega = \frac{1}{2}(1 + i_1 + i_2 + i_3), \quad i_3 = i_1 i_2 = -i_2 i_1, \; i_1^2 = i_2^2 = -1.$$

Freitag and Hermann [FH], 10.6 and 12.2, introduced 6 algebraically independent theta series on $H_2(\mathbb{H})$

$$(9) \qquad \sum_{g \in (i_1+i_2)^{-1}h + \Lambda^2} e^{2\pi i \overline{g}^{tr} Z g}, \quad Z \in H_2(\mathbb{H}), \quad h = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ \overline{\omega} \end{pmatrix}.$$

Setting $\sqrt{-2} = i_1 + i_2$ we obtain an embedding of $H_2(\mathbb{C})$ into $H_2(\mathbb{H})$ and of Γ_2 into the modular group of degree 2 over the Hurwitz quaternions (cf. [K3], [FH]). Denote the restrictions of the theta series (9) onto $H_2(\mathbb{C})$ by $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$. Thus

$$\Lambda = \mathcal{O} + \omega\mathcal{O}$$

yields a representation of each θ_j as a homogeneous polynomial of degree 2 in the functions of Θ. We have

$$\theta_5 = \theta_6$$

due to [FH], 11.1, as well as

$$\theta_j \in [\Gamma_2[\sqrt{-2}], 2, 1]^{sym}.$$

Note that

$$\Gamma_2/\Gamma_2[\sqrt{-2}] \cong Sp_2(\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6,$$

the symmetric group in 6 letters. Using MAGMA we now obtain a basis of the invariant polynomials in $\theta_1, \ldots, \theta_5$ by means of the theta transformation formula.

Theorem 6. *There are 5 algebraically independent symmetric Hermitian modular forms $H_k \in [\Gamma_2, k, 1]^{sym}$, $k = 4, 6, 8, 10, 12$, which are polynomials in $\theta_1, \ldots, \theta_5$. They are given by*

$$\begin{aligned}
H_4 &= \theta_1^2 + 3\theta_2^2 + 3\theta_3^2 + 3\theta_4^2 + 6\theta_5^2,\\
H_6 &= \theta_1(\theta_1^2 - 9\theta_2^2 - 9\theta_3^2 - 9\theta_4^2 - 36\theta_5^2) + 54\theta_2\theta_3\theta_4,\\
H_8 &= (\theta_1^2 - \theta_2^2 - \theta_3^2 - \theta_4^2 + \theta_5^2)\theta_5^2 - 2\theta_1\theta_2\theta_3\theta_4 + \theta_2^2\theta_3^2 + \theta_3^2\theta_4^2 + \theta_4^2\theta_2^2,\\
H_{10} &= \theta_1(\theta_1^2 - \theta_2^2 - \theta_3^2 - \theta_4^4 + 2\theta_5^2) - 2\theta_1(\theta_2^2\theta_3^2 + \theta_3^2\theta_4^2 + \theta_4^2\theta_2^2)\\
&\quad + \theta_2\theta_3\theta_4(\theta_1^2 + 3\theta_2^2 + 3\theta_3^2 + 3\theta_4^2 - 6\theta_5^2),\\
H_{12} &= (23\theta_1^2 - 3\theta_2^2 - 3\theta_3^2 - 3\theta_4^2 + 6\theta_5^2)\theta_5^4 + (2\theta_1^4 + 6\theta_2^4 + 6\theta_3^4 + 6\theta_4^4 + 12\theta_1\theta_2\theta_3\theta_4)\theta_5^2\\
&\quad - (\theta_2^2 + \theta_3^2 + \theta_4^2)(8\theta_1^2\theta_5^2 + 18\theta_1\theta_2\theta_3\theta_4) + 2\theta_1^3\theta_2\theta_3\theta_4 + 36\theta_2^2\theta_3^2\theta_4^2.
\end{aligned}$$

Proof. $H_k \in [\Gamma_2, k, 1]^{sym}$ follows from the construction. A calculation of the Fourier expansions and Igusa's result [Ig] show that $H_4|_{H_2(\mathbb{R})}, H_6|_{H_2(\mathbb{R})}, H_{10}|_{H_2(\mathbb{R})}, H_{12}|_{H_2(\mathbb{R})}$ generate the graded ring of Siegel modular forms of even weights and that $H_8|_{H_2(\mathbb{R})} \equiv 0$. This yields the algebraic independence. □

We add a few final

Remarks. a) One has $H_4 = E_4, H_6 = E_6 + \frac{45{,}505}{19} F_3^2$.
b) R is generated by $H_4, H_6, H_8, H_{10}, H_{12}$ and $\phi_3^2, \phi_3 F_5, F_5^2$. One may also replace $\phi_3 F_5$ by $f_8|_{H_2(\mathbb{C})}$ with f_8 from [K3].
c) Let $\theta = X_1 \cdot \ldots \cdot X_{10}$ be the product of the ten theta series in [FH], 10.3. Thus [FH], 11.9, shows that $0 \neq \theta|_{H_2(\mathbb{C})} \in [\Gamma_2, 20, 1]^{sym}$ vanishes on $\mathcal{H}_2 \cup \mathcal{H}_6$. Hence Corollary 2b shows that $\theta|_{H_2(\mathbb{C})}/\phi_{20}$ is a holomorphic Hermitian modular form of weight 0, i.e.

$$\theta|_{H_2(\mathbb{C})} = \alpha \cdot \phi_{20} \quad \text{for some } 0 \neq \alpha \in \mathbb{C}.$$

d) We conjecture that any $f \in [\Gamma_2, k, 1]^{sym}, k$ even, is a polynomial in the theta series (8) as proved for the analogous case $\mathbb{Q}(\sqrt{-3})$ in [DK]. This looks plausible due to the Molien series and Theorem 6. In principle a basis of the ring of invariant polynomials of the above representation can be calculated using Magma.

We thank Ingo Klöcker for his assistance in the computations.

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