The Maaß Space and Hecke Operators

by

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Abstract

We give a new proof of the fact that the Maaß space is invariant under all Hecke operators. It is based on the characterization of the Maaß space by a symmetry relation and certain commutation relations of the Hecke algebra for the Jacobi group.

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1 Introduction

The Maaß space, a special subspace of the space of Siegel modular forms of weight k with respect to the Siegel modular group Γ_2 is invariant with respect to all Hecke operators. This has been proven by Andrianov [A1] (cf. [EZ]). A Siegel modular form f belongs to the Maaß space if and only if the Fourier coefficients satisfy the so-called Maaß relations.

The proof by Andrianov is quite explicit. Andrianov calculates the action of the Hecke operators on the Fourier coefficients and finally shows that these new coefficients also satisfy the Maaß relations [M].

In this note we give a new proof based on additive symmetry properties of the Maaß space. We note that Borcherds products [B] can also be characterized by the analogous multiplicative symmetry properties [HM2]. We note that our proof has the potential to be transferred to Borcherds products. Consider the embedding

$$j : \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \hookrightarrow \mathrm{Sp}_2(\mathbb{R})$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

Let

$$\mathcal{M}(n) := \{M \in \mathbb{Z}^{2\times 2};\ \det M = n\}$$

and

$$M^* := \frac{1}{\sqrt{n}} \cdot M \quad \text{for } M \in \mathcal{M}(n).$$

Here $\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \mathcal{M}(1)$.

Let f be a Siegel modular form and $\underset{k}{|}$ the Petersson slash operator. Then we put for the identity matrix I

$$(1) \qquad f \underset{k}{|} T_\Sigma^{\uparrow}(n) : = \sum_{M:\Gamma\backslash\mathcal{M}(n)} f \underset{k}{|} j(M^*, I_2),$$

$$(2) \qquad f \underset{k}{|} T_\Sigma^{\downarrow}(n) : = \sum_{M:\Gamma\backslash\mathcal{M}(n)} f \underset{k}{|} j(I_2, M^*).$$

Simply exchanging the Σ-symbol by the Π-symbol leads to the operators $T_\Pi^{\uparrow}(n)$ and $T_\Pi^{\downarrow}(n)$.

Theorem 1. *Let f be a Siegel modular form of degree 2 and weight k.*

a) Then f belongs to the Maaß space if and only if

$$f \underset{k}{|} T_\Sigma^\uparrow(n) = f \underset{k}{|} T_\Sigma^\downarrow(n) \quad \text{for all } n \in \mathcal{C}_\Sigma = \mathbb{N}. \tag{3}$$

b) Then f is a Borcherds product if and only if there exists $\varepsilon(n,f) = \pm 1$ such that

$$f \underset{k}{|} T_\Pi^\uparrow(n) = \varepsilon(n,f) \cdot f \underset{k}{|} T_\Pi^\downarrow(n) \quad \text{for all } n \in \mathcal{C}_\Pi = \mathbb{N}. \tag{4}$$

Remark. a) Theorem 1 a) has been discovered and proven in [H2] and generalized to automorphic forms on the orthogonal group $\mathcal{O}(2,n)$ of signature $(2,n)$ in [HM1]. It is easy to see that the set $\mathcal{C}_\Sigma$ can be chosen to be equal to the set $\mathbb{P}$ of all prime numbers.

b) Variants of this results are given by Pitale and Schmidt [PS] and Heim [H3]. Hence it suffices if $\mathcal{C}_\Sigma$ consists of almost all primes. Applying Weissauer's results on the generalized Ramanujan-Petersson conjecture leads to $\mathcal{C}_\Sigma = \{p\}$ for any chosen prime number p (cf. [RS], [FPRS]).

The symmetric equation (3) has many interesting applications, but also the disadvantage that the involved operators are not really fitting in known setting of Hecke operators.

Surprisingly, breaking the symmetry by just applying the slash operator to diag $(1, \sqrt{p}, 1, 1/\sqrt{p})$ in (3) makes it possible to give an interpretation in the frame of the Hecke algebra $\mathcal{H}_2^J$, the Hecke algebra of the Jacobi group inside Γ_2, i.e. Klingen parabolic subgroup of Γ_2. Then we consider a well-known embedding of the Hecke algebra of Γ_2 into the Hecke algebra related to the Jacobi group. Now a commutation relation in the latter Hecke algebra yields the Hecke invariance of the Maaß space. A similar approach may apply to Borcherds products.

2 Notation

The space $\mathcal{M}_k(\Gamma_2)$ of Siegel modular forms of degree 2 and weight k consists of all holomorphic functions $f : \mathbb{H}_2 \to \mathbb{C}$ satisfying

$$f(Z) = f \underset{k}{|} M(Z) := \det(CZ+D)^{-k} f\big((AZ+B)(CZ+D)^{-1}\big)$$

for all Z in the Siegel half-space

$$\mathbb{H}_2 := \{Z = X + iY \in \mathbb{C}^{2\times 2};\ Z = Z^{tr}, Y > 0\}, \quad Z = \begin{pmatrix} \tau & z \\ z & \widetilde{\tau} \end{pmatrix},$$

and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the Siegel modular group

$$\Gamma_2 := \left\{ M \in \mathbb{Z}^{4\times 4};\ M^{tr} J M = J \right\}, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Each such f possesses a Fourier expansion of the form

$$f(Z) = \sum_{T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \geq 0} \alpha_f(T) e^{2\pi i \,\mathrm{trace}\,(TZ)}.$$

The *Maaß space* $\mathcal{M}_k^*(\Gamma_2)$ consists of all $f \in \mathcal{M}_k(\Gamma_2)$ satisfying

$$\alpha_f \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} = \sum_{d | gcd(n,r,m)} d^{k-1} \alpha_f \begin{pmatrix} nm/d^2 & r/2d \\ r/2d & 1 \end{pmatrix}$$

for all $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Now use (3) for a prime p. It was proved in [H2], [HM1] that $f \in \mathcal{M}_k(\Gamma_2)$ belongs to the Maaß space if and only if

$$\begin{aligned} & p^{k/2} \cdot f \begin{pmatrix} p\tau & \sqrt{p}z \\ \sqrt{p}z & \widetilde{\tau} \end{pmatrix} + p^{-k/2} \cdot \sum_{a \bmod p} f \begin{pmatrix} (\tau + a)/p & \sqrt{p}z \\ \sqrt{p}z & \widetilde{\tau} \end{pmatrix} \\ (5) \qquad & = p^{k/2} \cdot f \begin{pmatrix} \tau & \sqrt{p}z \\ \sqrt{p}z & p\widetilde{\tau} \end{pmatrix} + p^{-k/2} \cdot \sum_{a \bmod p} f \begin{pmatrix} \tau & \sqrt{p}z \\ \sqrt{p}z & (\widetilde{\tau} + a)/p \end{pmatrix} \end{aligned}$$

for all primes p and all $Z \in \mathbb{H}_2$ (cf. Theorem 1 and Remark 1). Note that (5) is equivalent to

$$\begin{aligned} & p^{k-1} \cdot \alpha_f \begin{pmatrix} n/p & r/2p \\ r/2p & m \end{pmatrix} + \alpha_f \begin{pmatrix} np & r/2 \\ r/2 & m \end{pmatrix} \\ = {} & p^{k-1} \cdot \alpha_f \begin{pmatrix} n & r/2p \\ r/2p & m/p \end{pmatrix} + \alpha_f \begin{pmatrix} n & r/2 \\ r/2 & mp \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}, \end{aligned}$$

whenever the Fourier coefficient is 0 if the matrix is not half-integral.

3 Embedding of Hecke algebras

Denote by

$$\mathcal{M}_2 := \{ M \in \mathbb{Z}^{4\times 4};\ M^{tr} J M = \nu J \text{ for some } \nu \in \mathbb{N} \}$$

the monoid of integral symplectic similitudes and by

$$\mathcal{H}_2 = \mathcal{H}(\Gamma_2, \mathcal{M}_2)$$

the associated Hecke algebra (cf. [A2], [F], [K]). Let

$$\Gamma_2^J := \left\{ M = \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_2 \right\},$$

$$\mathcal{M}_2^J := \left\{ M = \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & r \end{pmatrix} \in \mathcal{M}_2;\ r \in \mathbb{N} \right\},$$

stand for the associated Jacobi group inside Γ_2 resp. Jacobi monoid and by

$$\mathcal{H}_2^J := \mathcal{H}(\Gamma_2^J, \mathcal{M}_2^J)$$

the associated Jacobi-Hecke algebra. In view of

$$\Gamma_2^J \subset \Gamma_2, \quad \mathcal{M}_2 \subset \Gamma_2 \cdot \mathcal{M}_2^J \quad \text{and} \quad \Gamma_2 \cap \mathcal{M}_2^J \cdot (\mathcal{M}_2^J)^{-1} \subset \Gamma_2^J$$

the mapping

$$\sum_{M:\Gamma_2\backslash\mathcal{M}_2} t(\Gamma_2 M)\Gamma_2 M \mapsto \sum_{M:\Gamma_2^J\backslash\mathcal{M}_2^J} t(\Gamma_2 M)\Gamma_2^J M$$

induces an injective homomorphism of the Hecke algebras

$$\iota : \mathcal{H}_2 \to \mathcal{H}_2^J$$

(cf. [A2], Proposition 3.1.6, [K], I(6.4)).

We calculate the image for well-known generators of $\mathcal{H}_2$. Therefore let

$$\triangledown(p) := \sum_{a \bmod p} \Gamma_2^J \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & a \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \Gamma_2^J.$$

Lemma. *Given a prime p one has*

$$\begin{aligned}
\iota(\Gamma_2 \operatorname{diag}(1,1,p,p)\Gamma_2) &= \Gamma_2^J \operatorname{diag}(1,1,p,p)\Gamma_2^J + \Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J, \\
\iota(\Gamma_2(pI_4)\Gamma_2) &= \Gamma_2^J(pI_4)\Gamma_2^J, \\
\iota(\Gamma_2 \operatorname{diag}(1,p,p^2,p)\Gamma_2) &= \Gamma_2^J \operatorname{diag}(p,p^2,p,1)\Gamma_2^J + \Gamma_2^J \operatorname{diag}(p,1,p,p^2)\Gamma_2^J \\
&\quad + \Gamma_2^J \operatorname{diag}(1,p,p^2,p)\Gamma_2^J + \triangledown(p) - \Gamma_2^J(pI_4)\Gamma_2^J.
\end{aligned}$$

Proof. (cf. [H1] or [GN]) A straightforward calculation leads to the following right coset decompositions

$$\Gamma_2^J \operatorname{diag}(1,1,p,p)\Gamma_2^J = \dot{\bigcup_{b,c \bmod p}} \Gamma_2^J \begin{pmatrix} p & 0 & 0 & 0 \\ -c & 1 & 0 & b \\ & & 1 & c \\ & & 0 & p \end{pmatrix} \cup \dot{\bigcup_{\substack{S \in \mathrm{Sym}(2;\mathbb{Z}) \\ S \bmod p}}} \Gamma_2^J \begin{pmatrix} I & S \\ 0 & pI \end{pmatrix},$$

$$\Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J = \Gamma_2^J \operatorname{diag}(p,p,1,1) \cup \dot{\bigcup_{b \bmod p}} \Gamma_2^J \begin{pmatrix} 1 & 0 & b & 0 \\ 0 & p & 0 & 0 \\ & & p & 0 \\ & & 0 & 1 \end{pmatrix}.$$

The representatives on the right hand side are also representatives of $\Gamma_2 \backslash \Gamma_2 \operatorname{diag}(1,1,p,p)\Gamma_2$.

Finally observe that

$$\begin{aligned} &\Gamma_2^J \operatorname{diag}(1,p,p^2,p)\Gamma_2^J \\ &= \dot{\bigcup_{a \bmod p^2}} \Gamma_2^J \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & p & 0 & 0 \\ & & p^2 & 0 \\ & & 0 & p \end{pmatrix} \cup \dot{\bigcup_{c \bmod p}} \Gamma_2^J \begin{pmatrix} p^2 & 0 & 0 & 0 \\ -pc & p & 0 & 0 \\ & & 1 & c \\ & & 0 & p \end{pmatrix} \\ &\qquad \cup \dot{\bigcup_{\substack{b,c \bmod p \\ b \not\equiv 0 \bmod p}}} \Gamma_2^J \begin{pmatrix} p & 0 & b & bc \\ 0 & p & bc & bc^2 \\ & & p & 0 \\ & & 0 & p \end{pmatrix}, \end{aligned}$$

$$\Gamma_2^J \operatorname{diag}(p,1,p,p^2)\Gamma_2^J = \dot{\bigcup_{\substack{a,c \bmod p \\ b \bmod p^2}}} \Gamma_2^J \begin{pmatrix} p & 0 & 0 & pa \\ -c & 1 & 0 & b \\ & & p & pc \\ & & 0 & p^2 \end{pmatrix},$$

$$\Gamma_2^J \operatorname{diag}(p,p^2,p,1)\Gamma_2^J = \Gamma_2^J \operatorname{diag}(p,p^2,p,1),$$

$$\Gamma_2^J \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & a \\ & & p & 0 \\ & & 0 & p \end{pmatrix} \Gamma_2^J = \Gamma_2^J \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & a \\ & & p & 0 \\ & & 0 & p \end{pmatrix}, \quad a \in \mathbb{Z}.$$

Then the claim follows as above. □

4 Hecke operators

We introduce Hecke operators on Siegel modular forms $f \in \mathcal{M}_k(\Gamma_2)$ of degree 2 and weight k for a subgroup Γ of Γ_2 just as in [F] without any additional factors:

$$f \underset{k}{|} \Gamma M \Gamma := \sum_{N:\Gamma\backslash\Gamma M\Gamma} f \underset{k}{|} N, \quad M \in \mathcal{M}_2.$$

Theorem 2. *Given $f \in \mathcal{M}_k(\Gamma_2)$ the following assertions are equivalent:*

(i) $f \in \mathcal{M}_k^*(\Gamma_2)$.

(ii) For almost all primes p one has

$$f \underset{k}{|} \Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J = p^k \cdot f \underset{k}{|} \Gamma_2^J \operatorname{diag}(p,p^2,p,1)\Gamma_2^J + p^k \cdot f \underset{k}{|} \bigtriangledown(p).$$

(iii) There is a prime p satisfying

$$f \underset{k}{|} \Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J = p^k \cdot f \underset{k}{|} \Gamma_2^J \operatorname{diag}(p,p^2,p,1)\Gamma_2^J + p^k \cdot f \underset{k}{|} \bigtriangledown(p).$$

Proof. Note that

$$Z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix} Z \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix} = \begin{pmatrix} \tau & \sqrt{p}z \\ \sqrt{p}z & p\widetilde{\tau} \end{pmatrix} \tag{6}$$

is a bijection of $\mathbb{H}_2$. Now we apply (5) to the latter matrices instead of Z. Thus (5) becomes equivalent to

$$f \underset{k}{|} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} + \sum_{a \bmod p} f \underset{k}{|} \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & p & 0 & 0 \\ & & p & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$= p^k \cdot f \underset{k}{|} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^2 & 0 & 0 \\ & & p & 0 \\ & & 0 & 1 \end{pmatrix} + p^k \cdot \sum_{a \bmod p} f \underset{k}{|} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & a \\ & & p & 0 \\ & & 0 & p \end{pmatrix}.$$

Now insert the right coset decompositions of the Γ_2^J double cosets from the proof of the Lemma. Thus (ii) is equivalent with (5). Note that (ii) and (iii) become equivalent due to [FPRS]. The simpler equivalence of (i) and (ii) was already proved in [H3], [PS]. □

The mapping (6) destroys the symmetry in the p-Hecke relations. But this step allows us to interprete (5) as an identity among Hecke operators with respect to Γ_2^J. It is remarkable that on the other hand the bijection

$$Z \mapsto \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1 \end{pmatrix} Z \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1 \end{pmatrix}$$

does not allow us to write (5) in terms of Hecke operators for Γ_2^J.

The embedding ι in the Lemma says that for $f \in \mathcal{M}_k(\Gamma_2)$

$$f \underset{k}{|} \Gamma_2 \operatorname{diag}(1,1,p,p)\Gamma_2 = f \underset{k}{|} \Gamma_2^J \operatorname{diag}(1,1,p,p)\Gamma_2^J + f \underset{k}{|} \Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J$$

etc. In order to derive the invariance of the Maaß space under all Hecke operators it suffices to consider the standard generators and their embedding into $\mathcal{H}_2^J$. We quote [H1], Proposition 3.4, resp. [GN] or use a direct computation of the products according to the decompositions in the proof of the Lemma:

The elements in the Hecke algebra $\mathcal{H}_2^J$

$$\begin{array}{lll} \Gamma_2^J \operatorname{diag}(1,1,p,p)\Gamma_2^J, & \Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J, & \Gamma_2^J \operatorname{diag}(1,p,p^2,p)\Gamma_2^J, \\ \Gamma_2^J \operatorname{diag}(p,1,p,p^2)\Gamma_2^J, & \Gamma_2^J \operatorname{diag}(p,p^2,p,1)\Gamma_2^J, & \Gamma_2^J(pI_4)\Gamma_2^J, \quad \bigtriangledown(p) \end{array}$$

for a fixed prime p commute with all these elements for any prime q, $q \neq p$.

Note that $\mathcal{H}_2^J$ is not commutative and that a corresponding result does not hold for $q = p$. Thus the restriction to a almost all primes p in Theorem 2 is crucial for the new proof of the

Corollary 1. *The Maaß space $\mathcal{M}_k^*(\Gamma_2)$ is invariant under all Hecke operators from $\mathcal{H}_2$.*

Proof. Apply Theorem 1 and the Lemma. □

In his original proof Andrianov [A1] calculated the effect of the Hecke operators on the Fourier coefficients explicitly. As we are working with Γ_2^J it is possible to derive the action on the Fourier-Jacobi expansion of $f \in \mathcal{M}_k^*(\Gamma_2)$ explicitly just as pointed out in [GN].

5 Borcherds products

It was proved in [HM2] that the multiplicative version of (5) characterizes Borcherds products in $\mathcal{M}_k(\Gamma_2)$. If we introduce the multiplicative Hecke operator for $f \in \mathcal{M}_k(\Gamma_2)$ and a subgroup Γ of Γ_2 by

$$f \underset{k}{\sqcap} \Gamma M \Gamma := \prod_{N:\Gamma\backslash\Gamma M\Gamma} f \underset{k}{|} N \in \mathcal{M}_{\ell k}(\Gamma_2), \quad \ell := \sharp(\Gamma\backslash\Gamma M\Gamma),$$

the same arguments yield

Corollary 2. *Given $f \in \mathcal{M}_k(\Gamma_2)$ the following assertions are equivalent:*

(i) f is a Borcherds product.

(ii) For all primes p there exists an $\varepsilon(p,f) = \pm 1$ such that

$$f \underset{k}{\sqcap} \Gamma_2^J \operatorname{diag}(1,p,p,1)\Gamma_2^J = \varepsilon(p,f) \cdot p^{(p+1)k} f \underset{k}{\sqcap} \Gamma_2^J \operatorname{diag}(p,p^2,p,1)\Gamma_2^J$$
$$\cdot \prod_{a \bmod p} f \underset{k}{\sqcap} \Gamma_2^J \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & a \\ & & p & 0 \\ & & 0 & p \end{pmatrix} \Gamma_2^J.$$

References

[A1] Andrianov, A.N.: Modular decent and the Saito-Kurokawa conjecture. Invent. Math. **53**, 267-280 (1979).

[A2] Andrianov, A.N.: Quadratic Forms and Hecke Operators. Grundl. Math. Wiss. **286**, Springer-Verlag, Berlin-Heidelberg-New York (1987).

[B] Borcherds, R.E.: Automorphic forms with singularities on Grassmannians. Invent. Math. **132**, 491-562 (1998).

[EZ] Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Progress Math. **55**, Birkhäuser Verlag, Boston (1985).

[FPRS] Farmer, D., Pitale, A., Ryan, N., Schmidt, R.: Characterizations of the Saito-Kurokawa lifting. Rocky Mountain J. Math. **43**, 1747-1757 (2013).

[F] Freitag, E.: Siegelsche Modulfunktionen. Grundl. Math. Wiss. **254**, Springer-Verlag, Berlin-Heidelberg-New York (1983).

[GN] Gritsenko, V., Nikulin, V.: The Igusa modular forms and “the simplest” Lorentzian Kac-Moody algebras. Matem. Sbornik **187**, 1601-1643 (1996).

[H1] Heim, B.: Pullbacks of Eisenstein Series, Hecke-Jacobi Theory and Automorphic L-Functions. Proc. Symp. Pure Math. **66**, 201-238 (1999).

[H2] Heim, B.: On the Spezialschar of Maass. Int. J. Math. Math. Sci. **23** (2010).

[H3] Heim, B.: Distribution theorems for Saito-Kurokawa lifts. In Manickam, M. (ed.): Number Theory. Ramanujan Soc. Lect. Notes Series **15**, 31-41 (2011).

[HM1] Heim, B., Murase, A.: A characterization of the Maass space on $O(2, m+2)$ by symmetries. Int. J. Math. **23** (2012).

[HM2] Heim, B., Murase, A.: A characterization of holomorphic Borcherd lifts by symmetries. Preprint (2012).

[K] Krieg, A.: Hecke algebras. Mem. Am. Math. Soc. **435** (1990).

[M] Maaß, H.: Über eine Spezialschar von Modulformen zweiten Grades I, II, III. Invent. Math. **52**, 95-104; **53**, 249-253; **53**, 255-265 (1979).

[PS] Pitale, A., Schmidt, R.: Ramanujan-type results for Siegel cusp forms of degree 2, J. Raman. Math. Soc. **24**, 87-111 (2009).

[RS] Roberts, B., Schmidt, R.: On modular forms for the paramodular gruoup. In: Automorphic forms and zeta functions, Proc. Conf. Mem. Tsuneo Arakawa, World Scientific (2006).