§2. An example: $\mathbf{Z}_2$

$H = \mathbf{Z}_2 = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \}.$

<u>$d = 1$</u>

s acts on $k[x, y]$ by interchanging x and y.

${}^{H}k[x, y] = k[\tau, \delta]$ where $\tau = x + y$ and $\delta = xy$.

$\underline{d = 2}$

$x_1 \;\; x_2$

$y_1 \;\; y_2$

${}^{H}k[\mathrm{M}_{2,\,2}] = k[\tau_1, \tau_{2,} \delta_1, \delta_2, F_{12}]$

$\tau_1 = x_1 + y_{1,}\;\; \delta_1 = x_1 y_1$

$\tau_2 = x_2 + y_2$, a polarization of τ_1

$\delta_2 = x_2 y_2$, a polarization of δ_1

$F_{12} = x_1 x_2 + y_1 y_2$, a polarization of δ_1

Put $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then $g * x_1 = ax_1 + bx_2$ and $g * y_1 = ay_1 + by_2$.

$$g * \delta_1 = g * x_1 y_1 = (ax_1 + bx_2)(ay_1 + by_2) = a^2 x_1 y_1 + ab(x_1 y_2 + x_2 y_1) + b^2 x_2 y_2$$

$$x_1 x_2 + y_1 y_2 = (x_1 + y_1)(x_2 + y_2) - (x_1 y_2 + x_2 y_1)$$

<u>$d \geq 2$, char $k \neq 2$</u>

$x_1 \quad x_2 \; . \; . \; . \; x_d$

$y_1 \quad y_2 \; . \; . \; . \; y_d$

$^H k[\mathrm{M}_{2,\,d}] = \mathrm{GL}_d \ast {}^H k[\mathrm{M}_{2,\,2}] = k[\tau_i, \delta_i, F_{i\,j}]$

$\tau_i = x_i + y_i,\ 2 \leq i \leq d,$ polarization of τ_1

$\delta_i = x_i y_i,\ 2 \leq i \leq d,$ polarization of δ_1

$F_{i\,j} = x_i x_j + y_i y_j$, polarization of δ_1

$\underline{d = 3, \text{ char } k \neq 2}$

$^{H}k[\mathrm{M}_{2,\,3}] = k[\tau_1, \tau_{2,}\ \tau_{3,}\ \delta_1, \delta_2,\ \delta_3, F_{12}, F_{13}, F_{23}]$

$$F_{123} = x_1x_2x_3 + y_1y_2y_3$$

$2F_{123} = F_{12}\tau_3 + F_{13}\tau_2 + F_{23}\tau_1 - \tau_1\tau_2\tau_3$ in algebra of polarized invariants

$\underline{d = 3, \text{char } k = 2}$

$^{H}k[M_{2,3}] = k[\tau_1, \tau_2, \tau_3, \delta_1, \delta_2, \delta_3, F_{12}, F_{13}, F_{23}, F_{123}]$

$\tau_i = x_i + y_i$, $\delta_\mathrm{i} = x_i y_i$, $F_{ij} = x_i x_j + y_i y_j$

$\mathrm{GL}_3 * {}^{H}k[\mathrm{M}_{2,1}] = k[\tau_1, \tau_2, \tau_3, \delta_1, \delta_2, \delta_3, F_{12}, F_{13}, F_{23}]$

$F_{123} = x_1 x_2 x_3 + y_1 y_2 y_3$

$F_{123}{}^2 = F_{12}F_{13}F_{23} + \delta_1 \delta_2 \tau_3{}^2 + \delta_1 \delta_3 \tau_2{}^2 + \delta_2 \delta_3 \tau_1{}^2 \in \mathrm{GL}_3 * {}^{H}k[\mathrm{M}_{2,1}]$

<u>$d \geq 2$, char $k = 2$</u>

$^{H}k[\mathrm{M}_{2,\,d}] = k[\tau_i;\, \delta_i;\, F_{i\,j};\, \mu\cdot(x_1 \ldots x_i + y_1 \ldots y_i),\, \mu \in S_d]$

S_d: permutation group on d letters)

$$\mathrm{GL}_d * {}^{H}k[\mathrm{M}_{2,\,1}] = k[\tau_i;\, \delta_i;\, F_{i\,j}]$$

$$(\mu\cdot(x_1 \ldots x_i + y_1 \ldots y_i))^2 \in \mathrm{GL}_d * {}^{H}k[\mathrm{M}_{2,\,1}]$$

$\underline{d \geq 2, \text{ char } k = 2, H = \mathbf{Z}_2}$

$^{H}k[\mathrm{M}_{2,\,d}] = k[\tau_i;\, \delta_i;\, F_{i\,j};\, \mu\cdot(x_1 \ldots x_i + y_1 \ldots y_i),\, \mu \in S_d]$

$$\mathrm{GL}_d * {}^{H}k[\mathrm{M}_{2,\,1}] = k[\tau_i;\, \delta_i;\, F_{i\,j}]$$

$$(\mu\cdot(x_1 \ldots x_i + y_1 \ldots y_i))^2 \in \mathrm{GL}_d * {}^{H}k[\mathrm{M}_{2,\,1}]$$

(has generators of degree d)

<u>$d \geq 2$, char $k \neq 2$, $H = \mathbf{Z}_2$</u>

$x_1 \quad x_2 \ . \ . \ . \ x_d$

$y_1 \quad y_2 \ . \ . \ . \ y_d$

$^{H}k[\mathrm{M}_{2,\,d}] = \ \mathrm{GL}_d \ * \ ^{H}k[\mathrm{M}_{2,\,2}] = k[\tau_i, \delta_i, F_{i\,j}]$

$\tau_i = x_i + y_i$, $2 \leq i \leq d$, polarization of τ_1

$\delta_i = x_i y_i$, $2 \leq i \leq d$, polarization of δ_1

$F_{i\,j} = x_i x_j + y_i y_j$, polarization of δ_1

<u>$d \geq 2$, char $k = 2$, $H = \mathbf{Z}_2$</u>

$${}^{H}k[\mathrm{M}_{2,\,d}] = k[\tau_i;\ \delta_i;\ F_{i\,j};\ \mu{\cdot}(x_1 \ldots x_i + y_1 \ldots y_i),\ \mu \in S_d]$$

$$\mathrm{GL}_d \ast {}^{H}k[\mathrm{M}_{2,\,1}] = k[\tau_i;\ \delta_i;\ F_{i\,j}]$$

$$(\mu{\cdot}(x_1 \ldots x_i + y_1 \ldots y_i))^2 \in \mathrm{GL}_d \ast {}^{H}k[\mathrm{M}_{2,\,1}]$$